308 ROW REDUCTION EXAMPLES

JOSH SWANSON

ABSTRACT. This document summarizes two particular applications of Gauss-Jordan elimination.

1. HOLT, EXERCISE 1.4.26

We need to solve

$$\begin{aligned} 0a + 0b + 0c + d &= -3 \\ a + b + c + d &= 2 \\ 27a + 9b + 3c + d &= 5 \\ 64a + 16b + 4c + d &= 0. \end{aligned}$$

We apply Gauss-Jordan elimination to find the reduced row echelon form of the corresponding augmented matrix. We start with

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -3 \\ 1 & 1 & 1 & 1 & 2 \\ 27 & 9 & 3 & 1 & 5 \\ 64 & 16 & 4 & 1 & 0 \end{pmatrix}$$

Start by doing Gaussian elimination to get to echelon form. For that we first want a pivot in row 1, column 1. Before doing that, though, tt's convenient to do the following ERO's in order:

$$R_1 \Leftrightarrow R_4, \qquad R_1 - R_4 \Rightarrow R_1, \qquad R_2 - R_4 \Rightarrow R_2, \qquad R_3 - R_4 \Rightarrow R_3$$

resulting in

$$\begin{pmatrix} 64 & 16 & 4 & 0 & 3 \\ 1 & 1 & 1 & 0 & 5 \\ 27 & 9 & 3 & 0 & 8 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

It's now more convenient to have the second row on top before forming that first pivot: apply

$$R_1 \Leftrightarrow R_2, \qquad 64R_1 - R_2 \Rightarrow R_2, \qquad 27R_1 - R_3 \Rightarrow R_3$$

which results in

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 5 \\ 0 & 48 & 60 & 0 & 317 \\ 0 & 18 & 24 & 0 & 127 \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

Date: July 24, 2017.

To get to echelon form we can apply

$$R_3 - \frac{18}{48}R_2 \Rightarrow R_3$$

giving

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 5 \\ 0 & 48 & 60 & 0 & 317 \\ 0 & 0 & \frac{3}{2} & 0 & \frac{65}{12} \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

To get to reduced row echelon form apply

$$\frac{2}{3}R_3 \Rightarrow R_3, \qquad R_2 - 60R_3 \Rightarrow R_2, \qquad \frac{1}{48}R_2 \Rightarrow R_2, \qquad R_1 - R_3 \Rightarrow R_1, \qquad R_1 - R_2 \Rightarrow R_1$$

which results in

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & 0 & -\frac{1}{6} \\ 0 & 0 & 1 & 0 & \frac{65}{12} \\ 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

Consequently we read off the general solution of the original system as

$$\begin{aligned} a &= -\frac{1}{4} \\ b &= -\frac{1}{6} \\ c &= \frac{65}{12} \\ d &= -3. \end{aligned}$$

2. Holt, Exercise 1.4.24

We need to solve

$$\begin{aligned} 2a - b - 2c &= d \\ a + 3b + 12c &= d \\ 4a + 2b + 3c &= d \end{aligned}$$

Again apply Gauss-Jordan elimination to the augmented matrix. We start with

$$\begin{pmatrix} 2 & -1 & -2 & -1 & 0 \\ 1 & 3 & 12 & -1 & 0 \\ 4 & 2 & 3 & -1 & 0 \end{pmatrix}$$

Get a pivot in the first column by applying

$$2R_2 - R_1 \Rightarrow R_2, \qquad R_3 - 2R_1 \Rightarrow R_3$$

which results in

$$\begin{pmatrix} 2 & -1 & -2 & -1 & 0 \\ 0 & 7 & 26 & -1 & 0 \\ 0 & 4 & 7 & 1 & 0 \end{pmatrix}$$

Get a pivot in the second column, resulting in an echelon form matrix, by applying

$$7R_3 - 4R_2 \Rightarrow R_3, \qquad -\frac{1}{11}R_3 \Rightarrow R_3$$

which results in

$$\begin{pmatrix} 2 & -1 & -2 & -1 & 0 \\ 0 & 7 & 26 & -1 & 0 \\ 0 & 0 & 5 & -1 & 0 \end{pmatrix}$$

Start to put the matrix in reduced form by clearing out the column above the rightmost pivot, namely the 5, by applying

$$5R_2 - 26R_3 \Rightarrow R_2, \qquad 5R_1 + 2R_3 \Rightarrow R_1, \qquad \frac{1}{7}R_2 \Rightarrow R_2$$

which results in

$$\begin{pmatrix} 10 & -5 & 0 & -7 & 0 \\ 0 & 5 & 0 & 3 & 0 \\ 0 & 0 & 5 & -1 & 0 \end{pmatrix}$$

Zero out the entries above the pivot in the second column and clean things up by applying

$$R_1 + R_2 \Rightarrow R_1, \qquad \frac{1}{10}R_1 \Rightarrow R_1, \qquad \frac{1}{5}R_2 \Rightarrow R_2, \qquad \frac{1}{5}R_3 \Rightarrow R_3$$

which results in

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{5} & 0 \\ 0 & 1 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 \end{pmatrix}$$

Converting this back to a linear system, d is the only free variable, giving a general solution of

$$a = \frac{2}{5}d$$
$$b = -\frac{3}{5}d$$
$$c = \frac{1}{5}d.$$