Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 5 questions for a total of 47 points.
- You are allowed to have one 8.5” $\times$ 11” handwritten note sheet, both sides, and a TI-30X IIS basic scientific calculator
- Cheating will result in a zero and be reported to the Dean’s Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and *indicate that you have done so.*

Question	Points	Score
1	9	
2	9	
3	10	
4	12	
5	7	
Total:	47	

1. (a) (1 point) Expand $(A+B)(A-B)$.

Solution: Distributing gives

$$A(A-B)+B(A-B)=A^2-AB+BA-B^2.$$

(This is generally not A^2-B^2.)

(b) (1 point) What is nullity(0_{mn}), where 0_{mn} is the $m\times n$ matrix with all 0 entries?

Solution: Since $0_{mn}\vec{v}=\vec{0}$ for all $\vec{v}\in\mathbb{R}^n$, the null space is all of $\mathbb{R}^n$, so the nullity is n.

(c) (4 points) Define 2 of the following 3 terms: linear transformation, subspace, onto.

Solution:

- Linear transformation: a function $T\colon\mathbb{R}^n\to\mathbb{R}^m$ such that (a) $T(\vec{u}+\vec{v})=T(\vec{u})+T(\vec{v})$ for all $\vec{u},\vec{v}\in\mathbb{R}^n$ and (b) $T(c\vec{u})=cT(\vec{u})$ for all $\vec{u}\in\mathbb{R}^n$ and $c\in\mathbb{R}$.
- Subspace: a subset S in $\mathbb{R}^n$ where (a) if $\vec{u},\vec{v}\in S$, then $\vec{u}+\vec{v}\in S$; (b) if $\vec{u}\in S$ and $c\in\mathbb{R}$, then $c\vec{u}\in S$; and (c) $\vec{0}\in S$.
- Onto: a function $f\colon X\to Y$ is onto if for every $y\in Y$ there is at least one $x\in X$ for which $f(x)=y$.

(d) (3 points) Describe the linear transformation with matrix

$$\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}2&0\\0&3\end{pmatrix}=\begin{pmatrix}2&3\\0&3\end{pmatrix}$$

geometrically. (Pictures encouraged.)

Solution: (Pictures omitted.) The diagonal matrix corresponds to the linear transformation $T\colon\mathbb{R}^2\to\mathbb{R}^2$ which scales in the x-direction by a factor of 2 and scales in the y-direction by a factor of 3. The other matrix corresponds to a *shear* linear transformation $U\colon\mathbb{R}^2\to\mathbb{R}^2$ which adds $(y,0)$ to an input vector (x,y). The linear transformation whose matrix is the product is the composite $U\circ T$ which first scales using T and then shears using U.

2. (a) (4 points) Compute the inverse of the following matrix, or show it does not exist:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Solution: Row reduce by swapping rows to find

$$[A \mid I] = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = [I \mid A^{-1}],$$

so

$$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(This is an example of a *permutation matrix*. In fact, the inverse of a permutation matrix is its transpose in general.)

(b) (5 points) Express span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$ as the solution set of a linear system.

Solution: There are several approaches; here's one. The span is clearly two-dimensional and it lives in four-dimensional space, so we need to find $4 - 2 = 2$ "different" linear equations the vectors satisfy. By inspection, both satisfy

$$\begin{aligned} w - x - y + z &= 0 \\ w - 2x + z &= 0. \end{aligned}$$

Double-checking, the coefficient matrix of this linear system has (row) rank 2, so it has nullity $4 - 2 = 2$, so the two-dimensional span above is the entirety of the null space, as required.

3. (a) (3 points) Give an example of a matrix A where $A^6 = I_2$ yet no smaller (positive) power of A is I_2.

> **Solution:** Rotating counterclockwise by $2\pi/6 = 60°$ has corresponding matrix
> $$R(2\pi/6) = \begin{pmatrix} \cos(2\pi/6) & -\sin(2\pi/6) \\ \sin(2\pi/6) & \cos(2\pi/6) \end{pmatrix} = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

(b) (3 points) Give an example of a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ where $\ker T = \operatorname{range} T$.

> **Solution:** There are many examples. A simple one: let $T(\vec{e_1}) = T(\vec{e}_2) = \vec{0}$, $T(\vec{e}_3) = \vec{e}_1$, and $T(\vec{e}_4) = \vec{e}_2$. Here both the kernel and the range are $\operatorname{span}\{\vec{e}_1, \vec{e}_2\}$.

(c) (4 points) Give an example of linear transformations $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ and $U\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that (i) T is one-to-one, (ii) U is onto, and (iii) the composite $U \circ T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is neither one-to-one nor onto.

> **Solution:** To avoid confusion, write $\vec{e}_1, \vec{e}_2$ for the standard basis of $\mathbb{R}^2$ and $\vec{f_1}, \vec{f_2}, \vec{f_3}$ for the standard basis of $\mathbb{R}^3$. One example: let $T(\vec{e}_1) = \vec{f_1}$, $T(\vec{e}_2) = \vec{f_2}$, $U(\vec{f_1}) = \vec{0}$, $U(\vec{f_2}) = \vec{e}_1$, and $U(\vec{f_3}) = \vec{e}_2$. Now T is one-to-one, U is onto, but $(U \circ T)(\vec{e}_1) = \vec{0}$ so $U \circ T$ is not one-to-one. By the unifying theorem, $U \circ T$ is then also not onto; it's also easy to see directly that $\vec{e}_2 \notin \operatorname{range}(U \circ T)$.

4. In the following, assume you have access to a computer to perform computations like Gauss–Jordan elimination or standard matrix operations.

   (a) (6 points) Give a **step-by-step recipe** for determining if two sets of vectors span the same subspace of $\mathbb{R}^n$. Your answer should be detailed enough that your classmates could follow the recipe without asking you for more details.

Solution:

- Let A and B be the matrices whose *rows* are the two sets of vectors, respectively.
- Let E and F be the RREF's of A and B.
- Delete any zero rows from E and F.
  - If $E = F$, return **True**.
  - Otherwise, return **False**.

The idea is as follows. We encode the subspaces spanned by the two sets as rows spaces of matrices. Row reduction doesn't change the row space, and two RREF matrices have the same row space if and only if they are equal.

   (b) (6 points) You are given a basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ for $\mathbb{R}^3$ along with some additional vectors $\vec{b}_1, \vec{b}_2, \vec{b}_3 \in \mathbb{R}^4$. Give a **step-by-step recipe** for computing the matrix of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$ for which $T(\vec{u}_1) = \vec{b}_1$, $T(\vec{u}_2) = \vec{b}_2$, and $T(\vec{u}_3) = \vec{b}_3$.

Solution:

- Let A be the matrix whose *columns* are $\vec{u}_1, \vec{u}_2, \vec{u}_3$.
- Compute A^{-1}.
- Let B be the matrix whose *columns* are $\vec{b}_1, \vec{b}_2, \vec{b}_3$.
- Return BA^{-1}.

The idea is as follows. Let $\vec{c}_i$ be the ith column of the matrix of T. We have $\vec{c}_i = T(\vec{e}_i)$. Suppose we could write $\vec{e}_i = a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3$. Then

$$T(\vec{e}_i) = a_1T(\vec{u}_1)+a_2T(\vec{u}_2)+a_3T(\vec{u}_3) = a_1\vec{b}_1+a_2\vec{b}_2+a_3\vec{b}_3 = [\vec{b}_1\ \vec{b}_2\ \vec{b}_3]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = B\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

So, we want to compute a_1, a_2, a_3. Notice

$$\vec{e}_i = a_1\vec{u}_1 + a_2\vec{u}_2 + a_3\vec{u}_3 = [\vec{u}_1\ \vec{u}_2\ \vec{u}_3]\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = A\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.$$

A is invertible since its columns are a basis. Multiplying this last equation by A^{-1} and putting it all together, we have

$$\vec{c}_i = T(\vec{e}_i) = B\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = BA^{-1}\vec{e}_i.$$

5. A linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ has the following matrix A with reduced echelon form B:

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = B.$$

(a) (4 points) Give bases for **both** row(A) and col(A).

Solution: A basis for row(A) is given by the non-zero rows of B, giving

$$\left\{ \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix} \right\}.$$

A basis for col(A) is given by reading off the columns of A in which a pivot appears in B, giving

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

(b) (3 points) Is T one-to-one, onto, and/or invertible?

Solution: None of these. Since B has two pivots, it has rank 2, so the range of T is a two-dimensional subspace of $\mathbb{R}^3$, i.e. T is not onto. Since B has rank 2 and 4 columns, it has nullity $4-2=2$, so T has two-dimensional kernel and T is not one-to-one. Since T is neither onto nor one-to-one, T is not invertible.