Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 42 points.
- You are allowed to have one 8.5" $\times$ 11" handwritten note sheet, both sides, and a TI-30X IIS basic scientific calculator
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and *indicate that you have done so*.

| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 12 | |
| 3 | 8 | |
| 4 | 12 | |
| Total: | 42 | |

1. (a) (1 point) Write $\begin{pmatrix} \pi \\ 3 \\ 1 \\ 4 \end{pmatrix}$ as a linear combination of standard basis vectors.

**Solution:**

$$\pi\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3 + 4\vec{e}_4$$

(b) (3 points) (Check all that apply.) Which of the following statements are *always true*? Assume dimensions are compatible.

$\surd$ $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ $\surd$ $A(3\vec{x}) = 3(A\vec{x})$ ◯ if $A\vec{x} = \vec{b}$, then $\vec{x} = \frac{\vec{b}}{A}$
◯ $(a+b)(\vec{u}+\vec{v}) = a\vec{u} + b\vec{v}$ $\surd$ $[\vec{a}_1 \ \cdots \ \vec{a}_m]\vec{e}_1 = \vec{a}_1$

(c) (4 points) Define 2 of the following 3 terms: trivial linear combination; coefficient matrix; span.

**Solution:**

- Trivial linear combination: given vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$, the trivial linear combination of these vectors is

$$0\vec{v}_1 + \cdots + 0\vec{v}_m.$$

- Coefficient matrix: given a linear system with $n$ variables and $m$ equations, the corresponding coefficient matrix is the $m \times n$ matrix where the entry in row $i$, column $j$ is the coefficient on the $j$th variable in equation $i$.
- Span: the span of vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ is the set of all of their linear combinations,

$$\operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_m\} = \{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m \in \mathbb{R}^n : c_1, \ldots, c_m \in \mathbb{R}\}.$$

(d) (2 points) Recall that the set of unit vectors in $\mathbb{R}^3$ is

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Can the set of unit vectors in $\mathbb{R}^3$ be written as the span of some vectors? If so, what vectors? If not, why not?

**Solution:** The span of some vectors always includes the trivial linear combination which results in $\vec{0}$, but $\vec{0}$ is not a unit vector, so the set of unit vectors cannot be written as a span.

2. Consider the linear system

$$\begin{aligned} 3x_1 + 3x_2 + x_3 + 2x_4 + 2x_5 &= 0 \\ x_1 + \ \ x_2 + x_3 + 2x_4 + 2x_5 &= 0 \\ x_1 + \ \ x_2 \qquad + \ \ x_4 + 2x_5 &= 0 \end{aligned}$$

(a) (2 points) Write the linear system in the form $A\vec{x} = \vec{b}$.

**Solution:**

$$\begin{pmatrix} 3 & 3 & 1 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) (7 points) Give the general solution of the linear system.

**Solution:** The reduced echelon form of $A$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Columns 2 and 5 have no pivots, so we have free variables $x_2 = t_1$ and $x_5 = t_2$. Backsubstitution gives

$$\begin{aligned} x_1 &= -t_1 \\ x_2 &= t_1 \\ x_3 &= 2t_2 \\ x_4 &= -2t_2 \\ x_5 &= t_2 \end{aligned}$$

(c) (3 points) Express the solution set as the span of some vectors.

**Solution:** From (b) we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{pmatrix}$$

so the solution set is

$$\operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

3. Provide examples meeting the following specifications.

   (a) (2 points) An inconsistent linear system whose augmented matrix is in reduced echelon form.

   **Solution:** The simplest example is
   $$0x = 1$$
   which corresponds to the reduced echelon form augmented matrix
   $$\begin{pmatrix} 0 & 1 \end{pmatrix}.$$

   (b) (3 points) Vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ which are linearly *dependent* but where $\vec{u}_1$ is not in $\text{span}\{\vec{u}_2, \vec{u}_3\}$.

   **Solution:** One example is
   $$\vec{u}_1 = \vec{e}_1, \vec{u}_2 = \vec{e}_2, \vec{u}_3 = 2\vec{e}_2.$$
   The span of the last two is the $x_2$-axis, and $\vec{e}_1$ is not on the $x_2$-axis.

   (c) (3 points) Vectors $\vec{u}_1, \vec{u}_2, \vec{v}_1, \vec{v}_2 \in \mathbb{R}^3$ where the only vector in both $\text{span}\{\vec{u}_1, \vec{u}_2\}$ and $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is $\vec{0}$.

   **Solution:** One example is
   $$\vec{u}_1 = \vec{e}_1, \vec{u}_2 = 2\vec{e}_1, \vec{v}_1 = \vec{e}_2, \vec{v}_2 = \vec{e}_3.$$
   The span of $\vec{u}_1, \vec{u}_2$ is the $x$-axis, the span of $\vec{v}_1, \vec{v}_2$ is the $yz$-plane, and they only point in both of these is $\vec{0}$.

4. Describe *step-by-step* recipes for solving the following problems if you had access to a computer which could do basic operations like row reduction, vector arithmetic, etc. "Pseudocode" descriptions are encouraged. Do not simply provide a single example; describe the general procedure. **Briefly justify** your procedure.

   (a) (4 points) Given a linear system, describe how to determine if it has 0, 1, or infinitely many solutions.

**Solution:**

- Let $A$ be the augmented matrix of the linear system.
- Let $E$ be the reduced echelon form of $A$.
  - If $A$ has rows of the form $(0 \ \cdots \ 0 \ 1)$, return "0 solutions."
  - Otherwise, let $k$ be the number of columns without pivots, excluding the last column.
    - If $k = 0$, return "1 solution."
    - If $k > 0$, return "infinitely many solutions."

The idea is to row reduce the linear system into echelon form and either read off that it's inconsistent or read off the number of free variables, $k$. Zero free variables means a unique solution and at least one means infinitely many solutions.

   (b) (4 points) Given $\vec{u}_1, \ldots, \vec{u}_m \in \mathbb{R}^n$, describe how to determine if they span $\mathbb{R}^n$.

**Solution:**

- Let $A$ be the matrix whose $i$th *row* is $\vec{u}_i$.
- Let $E$ be the reduced echelon form of $A$.
  - If every *column* of $E$ has a pivot, return "spans $\mathbb{R}^n$."
  - Otherwise, return "does not span $\mathbb{R}^n$."

Row operations do not change the row space. We showed in class that if the $i$th column of $E$ does not have a pivot, then $\vec{e}_i$ is not in the row space, so it can't span $\mathbb{R}^n$. Otherwise, removing zero rows from $E$, it's easy to see the $i$th row is $\vec{e}_i$ itself, so the rows must span $\mathbb{R}^n$.

Another approach:

- Let $B$ be the matrix whose $i$th *column* is $\vec{u}_i$.
- Let $E$ be the reduced echelon form of $B$.
  - If every *row* of $E$ has a pivot, return "spans $\mathbb{R}^n$."
  - Otherwise, return "does not span $\mathbb{R}^n$."

The vectors span $\mathbb{R}^n$ if and only if $[\vec{u}_1 \ \cdots \ \vec{u}_m]\vec{x} = \vec{b}$ has a solution for all $\vec{b}$. Computing the echelon form of $B$ is essentially row reducing this linear system; the details are carried out in the proof of Theorem 2.8 in Holt.

(c) (4 points) Given two linear systems with the same variables, describe how to determine whether or not they have the same solution set. (Hint: be careful about the number of equations and inconsistent systems.)

**Solution:**

- Using (a), determine if each of the two systems is consistent.
  - If both are inconsistent, return "same solution set."
  - If only one is inconsistent, return "different solution set."
  - If both are consistent, continue.
- Let $A$ and $B$ be the augmented matrix of the linear systems.
- Compute the *reduced* echelon forms $E$ and $F$ of $A$ and $B$, respectively.
- Delete all zero rows from both $E$ and $F$.
  - If $E = F$, return "same solution sets."
  - If $E \neq F$, return "different solution sets."

The idea is that solution sets essentially correspond with reduced echelon form matrices. There are some caveats to deal with: we could add zero rows to a reduced echelon form matrix without changing the solution set of the underlying linear system; and two inconsistent linear systems could have very different reduced echelon form augmented matrices. The above steps take care of these caveats and afterwards simply compare the underlying reduced echelon form matrices.