Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- Check that you have a complete exam. There are 8 questions for a total of 100 points.
- You are allowed to have one 8.5” × 11” handwritten note sheet, both sides, and a TI-30X IIS basic scientific calculator.
- Cheating will result in a zero and be reported to the Dean’s Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and *indicate that you have done so.*

Question	Points	Score
1	13	
2	14	
3	16	
4	12	
5	14	
6	9	
7	12	
8	10	
Total:	100	

1. (a) (3 points) Compute

$$\det \begin{pmatrix} 0 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix}.$$

Solution: Expand along the first column, then the first column of the resulting submatrix, giving

$$-4 \cdot 2 \cdot \det \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} = -8.$$

(b) (6 points) Suppose A is a square matrix. Let $X = \{\vec{x} : A\vec{x} = A^T\vec{x}\}$. Is X a subspace? If so, verify it. If not, give an example justifying why not.

Solution: Since $A\vec{x} = A^T\vec{x}$ if and only if $(A - A^T)\vec{x} = \vec{0}$, we see that X is $\text{null}(A - A^T)$, and null spaces are subspaces. Alternatively, we may verify the axioms directly:

- $\vec{0} \in X$ since $A\vec{0} = \vec{0} = A^T\vec{0}$.
- If $\vec{u}, \vec{v} \in X$, then $A\vec{u} = A^T\vec{u}$ and $A\vec{v} = A^T\vec{v}$. Adding these equations gives $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = A^T\vec{u} + A^T\vec{v} = A^T(\vec{u} + \vec{v})$, so $\vec{u} + \vec{v} \in X$.
- If $\vec{u} \in X$ and $c \in \mathbb{R}$, then $A\vec{u} = A^T\vec{u}$. Multiplying this equation by c and using properties of matrix arithmetic gives $A(c\vec{u}) = A^T(c\vec{u})$, so $c\vec{u} \in X$.

(c) (4 points) Define **two** of the following three terms: eigenvector, transpose, basis.

Solution:

- Eigenvector: given a square matrix A, an eigenvector of A is a non-zero vector $\vec{v}$ where $A\vec{v} = \lambda\vec{v}$ and λ is some scalar.
- Transpose: given an $n \times m$ matrix A, its transpose A^T is the $m \times n$ matrix where $(A^T)_{ij} = A_{ji}$.
- Basis: given a subspace S, a basis for S is a linearly independent subset of S which spans S.

2. Give examples matching the following specifications. You **do not** need to justify your answers for this question.

   (a) (2 points) A two-dimensional subspace of $\mathbb{R}^4$.

   > **Solution:** The x_1, x_2-plane, $\operatorname{span}\{\vec{e}_1, \vec{e}_2\}$, is one of many answers.

   (b) (3 points) Two different 3×4 matrices in RREF with no zero rows and the same set of pivot positions.

   > **Solution:** There are many answers. One is
   >
   > $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (c) (5 points) Two matrices which are *clearly similar* but which do not commute, i.e. $AB \neq BA$.

   > **Solution:** Let's try $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is its own inverse, and $A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. Set
   >
   > $$B = PAP^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix},$$
   >
   > which is similar to A by definition. It is easy to check directly that $AB \neq BA$ in this example. "Almost all" such examples will end up working.

   (d) (4 points) A linear transformation T where

   $$\operatorname{range} T = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \ker T = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

   > **Solution:** The kernel is the span of $\vec{e}_1, \vec{e}_3$. So, we can let $T(\vec{e}_1) = T(\vec{e}_3) = \vec{0} \in \mathbb{R}^3$ and $T(\vec{e}_2) = \vec{v}_1$, $T(\vec{e}_4) = \vec{v}_2$ where $\vec{v}_1$, $\vec{v}_2$ are the above vectors. This has matrix
   >
   > $$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which has characteristic polynomial $\lambda^2(\lambda-1)(\lambda-3)$.

(a) (2 points) What are the eigenvalues of A?

Solution: The roots of the characteristic polynomial are $0, 1, 3$.

(b) (2 points) What are the algebraic multiplicities of the eigenvalues of A?

Solution: The multiplicities of the above roots of the characteristic polynomial are $2, 1, 1$, respectively.

(c) (4 points) Give a basis for the 0-eigenspace.

Solution: We need to compute a basis for null(A). Putting A into RREF gives

$$\begin{pmatrix} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Reading off the general solution of the corresponding homogeneous linear system gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -s/3 \\ 2s/3 \\ s \\ 0 \end{pmatrix}$$

so the null space is 1-dimensional with basis $\left\{ \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \\ 0 \end{pmatrix} \right\}$

(d) (2 points) What are the geometric multiplicities of the eigenvalues of A?

Solution: They are all 1. The eigenvalues $1, 3$ have geometric multiplicity at least 1, their algebraic multiplicities are 1, and geometric multiplicities in general are no larger than algebraic multiplicities. Hence the eigenvalues $1, 3$ have geometric multiplicity exactly 1. The previous part showed the eigenvalue 0 also has geometric multiplicity 1.

(e) (2 points) Is A diagonalizable?

Solution: No. Diagonalizable matrices have equal geometric and algebraic multiplicities of all eigenvalues, whereas here the algebraic and geometric multiplicities of $\lambda = 0$ are 2 and 1, respectively.

(f) (2 points) Is A invertible?

Solution: No. One of the conditions in the unifying theorem for a matrix to be invertible is that 0 is not an eigenvalue, but here 0 is an eigenvalue. Also, the fourth row is zero.

(g) (2 points) What are the rank and nullity of A?

Solution: The nullity is the geometric multiplicity of the 0-eigenspace, which is 1 as above. By the rank-nullity theorem, the rank must then be $4 - 1 = 3$.

4. (a) (6 points) Suppose the characteristic polynomial of A is λ^2+1. What is the characteristic polynomial of A^2? (There is in fact enough information to answer this question in this particular case.)

Solution: Using the fact that there is enough information, we can just find an example and compute its square's characteristic polynomial. Tinkering around, the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}$$

has characteristic polynomial λ^2+1. Its square has characteristic polynomial $\lambda^2+2\lambda+1$, so this is the answer.

One way to see that there is in fact enough information is the following. Given a general matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the characteristic polynomial is $\lambda^2 - (a+d)\lambda + (ad-bc)$. Hence we have $ad - bc = 1$ and $a = -d$. The square of A similarly has characteristic polynomial $\lambda^2 - (a^2 + 2bc + d^2)\lambda + (ad-bc)^2$. Here $(ad-bc)^2 = 1^2 = 1$ and $a^2+2bc+d^2 = a^2+2(ad-1)+(-a)^2 = 2a^2+2a(-a)-2 = -2$. Thus the characteristic polynomial of A^2 is indeed $\lambda^2+2\lambda+1$.

(b) (3 points) Suppose A and B are similar matrices. If A is invertible, is B invertible?

Solution: Yes. We have $A = PBP^{-1}$ for some P, so $B = P^{-1}AP$. We then have $B^{-1} = P^{-1}A^{-1}P$ since $B(P^{-1}A^{-1}P) = (P^{-1}AP)(P^{-1}A^{-1}P) = \cdots = I$.

(c) (3 points) Suppose A and B are similar and invertible. Is A^{-1} similar to B^{-1}?

Solution: Yes. As above, we have $B^{-1} = P^{-1}A^{-1}P$, so $PB^{-1}P^{-1} = A^{-1}$, so A^{-1} and B^{-1} are similar.

5. Let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$
Hint: the eigenvalues of A are $\frac{1\pm\sqrt{5}}{2}$. The identity $\frac{2}{1\mp\sqrt{5}} = -\frac{1\pm\sqrt{5}}{2}$ may be useful.

(a) (10 points) Diagonalize the matrix A. That is, write $A = PDP^{-1}$ where D is diagonal.

Solution: We must compute a basis of eigenvectors. We find
$$A - \frac{1\pm\sqrt{5}}{2} I = \begin{pmatrix} \frac{1\mp\sqrt{5}}{2} & 1 \\ 1 & -\frac{1\pm\sqrt{5}}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1\pm\sqrt{5}}{2} \\ 0 & 0 \end{pmatrix}$$
so a basis for the $\frac{1\pm\sqrt{5}}{2}$-eigenspace is $\left\{ \begin{pmatrix} \frac{1\pm\sqrt{5}}{2} \\ 1 \end{pmatrix} \right\}$. Consequently,
$$A = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1}.$$

(b) (4 points) What is $A^{1000}\vec{e}_1$? Give an explicit expression in terms of the eigenvalues of A.

Solution: Using $A = PDP^{-1}$ above, we have $A^{1000} = PD^{1000}P^{-1}$. Hence
$$A^{1000}\vec{e}_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{1000} & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \cdots = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{1000} - \left(\frac{1-\sqrt{5}}{2}\right)^{1000} \right).$$
It turns out the second exponential is essentially zero, so the first exponential is the main contribution. (This is incidentally the 1000th Fibonacci number and this computation proves "Binet's formula.")

6. (a) (3 points) Give an example of a 3×3 projection matrix. Give a basis for the subspace it is projecting onto.

Solution: There are many examples. An easy one is to project onto the xy-plane. The corresponding matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis for the subspace is $\{\vec{e}_1, \vec{e}_2\}$.

(b) (6 points) Recall that

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the matrix of the linear transformation rotating counterclockwise by a fixed angle θ about the origin in $\mathbb{R}^2$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which rotates its input by a fixed angle θ counterclockwise about the positive z-axis in $\mathbb{R}^3$. What is the matrix of T?

Solution: For vectors in the xy-plane, we can essentially just apply $R(\theta)$ to the x and y coordinates and keep the z-coordinate zero. Hence we see $T(\vec{e}_1)$ and $T(\vec{e}_2)$ are

$$\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}.$$

We also see $T(\vec{e}_3) = \vec{e}_3$. The matrix of T is hence

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

7. In the following, assume you have access to a computer to perform computations like Gauss–Jordan elimination or standard matrix operations. In this question you do not need to justify why your procedures work.

   (a) (6 points) Give a **step-by-step recipe** for determining if a linear transformation is one-to-one. Your answer should be detailed enough that your classmates could follow your recipe without asking you for more details. For this problem, suppose you're able to compute values of the linear transformation on particular inputs.

   Solution: Let T be the linear transformation.

   - Let A be the matrix whose ith column is $T(\vec{e}_i)$.
   - Let B be the RREF of A.
   - Let k be the number of columns of B without pivots.
     - If $k > 0$, return **NOT ONE-TO-ONE**.
     - If $k = 0$, return **ONE-TO-ONE**.

   Here k is the nullity of A.

   (b) (6 points) Suppose you've been given $n-1$ linearly independent vectors in $\mathbb{R}^n$. Give a **step-by-step recipe** for finding a vector which, when added to the others, yields a basis for $\mathbb{R}^n$.

   Solution: There are many solutions; one is the following.

   - Let $i = 1$.
   - Let A be the $n \times n$ matrix whose first $n-1$ columns are the given vectors and whose nth column is $\vec{e}_i$.
   - Compute $\det(A)$.
     - If $\det(A) \neq 0$, return $\vec{e}_i$.
     - Otherwise, increment i and repeat the above procedure.

   As described, it's not clear the process terminates. If $\vec{e}_i$ fails the above test, then $\vec{e}_i$ is in the span of the $n-1$ vectors. Hence the above process can't fail for all i since if it did then the $n-1$ vectors would span $\mathbb{R}^n$.

8. (a) (5 points) Suppose A is a 5×5 matrix and a basis for the column space of A has 3 vectors. Can a basis for the column space of A^2 contain 4 vectors? Explain your answer.

Solution: No, it cannot contain 4 vectors. From the definition of matrix multiplication, the columns of A^2 are linear combinations of the columns of A. Hence a basis for the columns of A must span the columns of A^2. Thus the column space of A^2 has a spanning set with 3 vectors, so any basis can have no more than 3 vectors.

(b) (5 points) Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

be bases for $\mathbb{R}^2$. Suppose $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. What are $\vec{x}$ and $[\vec{x}]_{\mathcal{C}}$?

Solution: Unpacking the definition of coordinate vectors, we have

$$\vec{x} = 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

We could use a change of basis matrix to convert this to the $\mathcal{C}$ basis, or we could observe directly that

$$\vec{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

so that

$$[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$