Math 308 F **Final** **Spring 2018**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- Check that you have a complete exam. There are 8 questions for a total of 100 points.
- You are allowed to have one 8.5" × 11" handwritten note sheet, both sides, and a TI-30X IIS basic scientific calculator.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and *indicate that you have done so.*

Question	Points	Score
1	13	
2	14	
3	16	
4	12	
5	14	
6	9	
7	12	
8	10	
Total:	100	

1. (a) (3 points) Compute

$$\det \begin{pmatrix} 0 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 4 & 3 \end{pmatrix}.$$

(b) (6 points) Suppose A is a square matrix. Let $X = \{\vec{x} : A\vec{x} = A^T\vec{x}\}$. Is X a subspace? If so, verify it. If not, give an example justifying why not.

(c) (4 points) Define **two** of the following three terms: eigenvector, transpose, basis.

2. Give examples matching the following specifications. You **do not** need to justify your answers for this question.

   (a) (2 points) A two-dimensional subspace of $\mathbb{R}^4$.

   (b) (3 points) Two different 3×4 matrices in RREF with no zero rows and the same set of pivot positions.

   (c) (5 points) Two matrices which are *clearly similar* but which do not commute, i.e. $AB \neq BA$.

   (d) (4 points) A linear transformation T where

$$\operatorname{range} T = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \ker T = \left\{ \begin{pmatrix} x_1 \\ 0 \\ x_2 \\ 0 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

3. Let $A = \begin{pmatrix} 3 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which has characteristic polynomial $\lambda^2(\lambda - 1)(\lambda - 3)$.

   (a) (2 points) What are the eigenvalues of A?

   (b) (2 points) What are the algebraic multiplicities of the eigenvalues of A?

   (c) (4 points) Give a basis for the 0-eigenspace.

   (d) (2 points) What are the geometric multiplicities of the eigenvalues of A?

   (e) (2 points) Is A diagonalizable?

(f) (2 points) Is A invertible?

(g) (2 points) What are the rank and nullity of A?

4. (a) (6 points) Suppose the characteristic polynomial of A is $\lambda^2 + 1$. What is the characteristic polynomial of A^2? (There is in fact enough information to answer this question in this particular case.)

(b) (3 points) Suppose A and B are similar matrices. If A is invertible, is B invertible?

(c) (3 points) Suppose A and B are similar and invertible. Is A^{-1} similar to B^{-1}?

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hint: the eigenvalues of A are $\frac{1\pm\sqrt{5}}{2}$. The identity $\frac{2}{1\mp\sqrt{5}} = -\frac{1\pm\sqrt{5}}{2}$ may be useful.

(a) (10 points) Diagonalize the matrix A. That is, write $A = PDP^{-1}$ where D is diagonal.

(b) (4 points) What is $A^{1000}\vec{e}_1$? Give an explicit expression in terms of the eigenvalues of A.

6. (a) (3 points) Give an example of a 3×3 projection matrix. Give a basis for the subspace it is projecting onto.

(b) (6 points) Recall that

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the matrix of the linear transformation rotating counterclockwise by a fixed angle θ about the origin in $\mathbb{R}^2$. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation which rotates its input by a fixed angle θ counterclockwise about the positive z-axis in $\mathbb{R}^3$. What is the matrix of T?

7. In the following, assume you have access to a computer to perform computations like Gauss–Jordan elimination or standard matrix operations. In this question you do not need to justify why your procedures work.

   (a) (6 points) Give a **step-by-step recipe** for determining if a linear transformation is one-to-one. Your answer should be detailed enough that your classmates could follow your recipe without asking you for more details. For this problem, suppose you're able to compute values of the linear transformation on particular inputs.

   (b) (6 points) Suppose you've been given $n-1$ linearly independent vectors in $\mathbb{R}^n$. Give a **step-by-step recipe** for finding a vector which, when added to the others, yields a basis for $\mathbb{R}^n$.

8. (a) (5 points) Suppose A is a 5×5 matrix and a basis for the column space of A has 3 vectors. Can a basis for the column space of A^2 contain 4 vectors? Explain your answer.

(b) (5 points) Let

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{C} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

be bases for $\mathbb{R}^2$. Suppose $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. What are $\vec{x}$ and $[\vec{x}]_{\mathcal{C}}$?