Math 308 M – Spring 2015
Proof Homework 3
Due Monday, June 1st, 2015

Name: ______________________________

- **Answer one** of the following two questions. Only one problem will be graded.
- Give rigorous proofs. Any skipped steps must be small enough that you could explain them to me in a few seconds. Your goal is to convince me you fully understand your argument and have not missed anything.
- You may use any theorem, proposition, etc. from lecture or the book, though when you do say at least "from the book" or "from lecture."
- For examples to model your proofs on, see the textbook, the proof examples document on the course web site, or any of the alternatives to the textbook linked from the course web site.
- You are welcome to talk to others (even outside the class) or work in groups on this assignment, though **write your final answers alone**. Keep in mind that this exercise is entirely for your benefit in becoming more comfortable with proofs.

1. Let $S_1$ and $S_2$ be subspaces of $\mathbb{R}^n$.

   Recall the *intersection* of two sets is the set of elements in both sets. For instance, in $\mathbb{R}^2$, the intersection of $\text{Span}\{\mathbf{e}_1\}$ and $\text{Span}\{\mathbf{e}_2\}$ is the intersection of the $x$ and $y$-axes, namely the origin. In symbols,

$$\text{Span}\{\mathbf{e}_1\} \cap \text{Span}\{\mathbf{e}_2\} = \{\mathbf{0}\}.$$

   (a) Suppose $S_1$ and $S_2$ are subspaces of $\mathbb{R}^n$. Show that the intersection $S_1 \cap S_2$ is a subspace of $\mathbb{R}^n$.

   (b) Suppose $S_1$ and $S_2$ are subspaces of $\mathbb{R}^n$ and that $S_1 \cap S_2 = \{\mathbf{0}\}$. Let $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_i\}$ be a basis for $S_1$ and let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_j\}$ be a basis for $S_2$. Show that the union $U \cup V = \{\mathbf{u}_1, \ldots, \mathbf{u}_i, \mathbf{v}_1, \ldots, \mathbf{v}_j\}$ is linearly independent.

2. Let $A$ and $B$ be $n \times n$ matrices.
   (a) Suppose $A^2 = A$. Show that $\det(A)$ is either 1 or 0.

   (b) Show $\operatorname{Tr}(A + B) = \operatorname{Tr}(A) + \operatorname{Tr}(B)$ directly from the definition.

   (c) Show $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ directly from the definition. (Sigma notation might be useful.)

   (d) Use the previous part to show that $\operatorname{Tr}(ABC) = \operatorname{Tr}(BCA)$. (Hint: recall that $ABC = (AB)C$. Warning: $\operatorname{Tr}(ABC) \neq \operatorname{Tr}(CBA)$ in general—only "cyclic rotations" work in general.)