Math 308 M – Spring 2015 Proof Homework 2 Due Wednesday, May 6th, 2015

Name: _____

- **Answer one** of the following two questions. If you answer them both, your *better* answer will be ignored.
- Give rigorous proofs. Any skipped steps must be small enough that you could explain them to me in a few seconds. Your goal is to convince me you fully understand your argument and have not missed anything.
- You may use any theorem, proposition, etc. from lecture or the book, though when you do say at least "from the book" or "from lecture."
- For examples to model your proofs on, see the textbook, the proof examples document on the course web site, or any of the alternatives to the textbook linked from the course web site.
- You are welcome to talk to others (even outside the class) or work in groups on this assignment, though **write your final answers alone**. Keep in mind that this exercise is entirely for your benefit in becoming more comfortable with proofs.

- 1. Let X be a 2×2 matrix with trace 0.
 - (a) Find constants α, β, γ with $\alpha \neq 0$ (which depend on the matrix X) so that

$$\alpha X^2 + \beta X + \gamma I = 0.$$

(Here I denotes the 2×2 identity matrix and 0 denotes the 2×2 zero matrix,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.)$$

Solution: Suppose

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By assumption a + d = 0, so replace d with -a. Compute the square:

$$X^{2} = \begin{bmatrix} a^{2} + bc & 0\\ 0 & a^{2} + bc \end{bmatrix} = (a^{2} + bc) \begin{bmatrix} 1 & 0\\ 0 & 1 \end{bmatrix}.$$

Hence

$$X^2 - (a^2 + bc)I = 0,$$

so $\alpha = 1, \beta = 0, \gamma = -(a^2 + bc)$ works. There are other solutions (for instance, double each of α, β, γ to get another one), but we only needed to find one.

(b) Use the result of Proof Homework 1, Problem 2 and your answer from (a) to show that the columns of X are linearly dependent if and only if $X^2 = 0$.

Solution: We continue the notation from (a). Proof Homework 1, Problem 2 says that $X\mathbf{u} = \mathbf{0}$ has more than one solution if and only if $-a^2 = bc$. From the definition of matrix-column multiplication, $X\mathbf{u} = \mathbf{0}$ has more than one solution if and only if the columns of X are linearly dependent. That is, $-a^2 = bc$ if and only if the columns of X are linearly dependent.

On the other hand, from (a) we have $X^2 = (a^2 + bc)I$, so $X^2 = 0$ if and only if $a^2 + bc = 0$, so if and only if $-a^2 = bc$. In all, $X^2 = 0$ if and only if the columns of X are linearly dependent.

- 2. Let X be an $n \times n$ matrix.
 - (a) Suppose that $X^{100} = 0$. Show that the columns of X are linearly dependent.

Solution: There are many solutions. Here's three.

- (1) Proof by contradiction: suppose to the contrary the columns of X are linearly independent. By the Big Theorem, X is an invertible matrix. Since the product of invertible matrices is invertible, $X^{100} = 0$ is invertible, However, the zero matrix is clearly not invertible, a contradiction. Hence the columns of X must be linearly dependent.
- (2) Proof by contradiction: if the columns are linearly independent, the linear transformation $T(\mathbf{u}) := X\mathbf{u}$ is one-to-one. Note that $(T \circ T \circ \cdots \circ T)(\mathbf{u}) = T^{100}(\mathbf{u}) = X^{100}\mathbf{u} = \mathbf{0}$ for all \mathbf{u} , so T^{100} is not at all one-to-one. However, the composite of one-to-one functions is one-to-one, so T^{100} is one-to-one, a contradiction. We verify this last claim now. If $f: U \to V$ and $g: V \to W$ are one-to-one, then $g \circ f: U \to W$ is one-to-one, since if $(g \circ f)(u_1) = (g \circ f)(u_2)$, then $g(f(u_1)) = g(f(u_2))$, so $f(u_1) = f(u_2)$ since g is one-to-one, but then $u_1 = u_2$ since f is one-to-one.
- (3) A different, direct, and algorithmic solution: From the definition of matrix multiplication, $X\mathbf{u}$ is a linear combination of the columns of X where the scale factors are the entries of \mathbf{u} . Hence a non-zero solution \mathbf{u} to $X\mathbf{u} = \mathbf{0}$ is a "certificate" of linear dependence.

Now $X^{100}\mathbf{v} = X(X^{99}\mathbf{v}) = \mathbf{0}$, so if $X^{99}\mathbf{v} \neq \mathbf{0}$, we may use $\mathbf{u} = X^{99}\mathbf{v}$ as our certificate. If not, we at least have $X^{99}\mathbf{v} = \mathbf{0}$. But this says $X(X^{98}\mathbf{v}) = \mathbf{0}$. If $X^{98}\mathbf{v} \neq \mathbf{0}$, we may use $\mathbf{u} = X^{98}\mathbf{v}$ as our certificate. If not, Continuing in this manner, in the "worst case" we end up with $X^{1}\mathbf{v} = \mathbf{0}$, so $\mathbf{u} = \mathbf{v}$ can be our certificate, at least assuming we had chosen \mathbf{v} non-zero to start.

(b) Show that the converse of (a) does not hold. That is, find a matrix X whose columns are linearly dependent but where no power of X is the zero matrix.

(Note: 1(b) can be interpreted as saying your counterexample cannot be a 2×2 trace 0 matrix.)

Solution: There are many, many examples, and virtually any "arbitrarily chosen" matrix with linearly dependent columns will have this property. Perhaps the simplest example comes from "projection matrices", say

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Geometrically, this projects a point (x, y) onto the x-axis, namely it sends (x, y) to (x, 0). Repeating this geometric procedure arbitrarily many times does

nothing more than applying it once, so $X^k = X$ for all $k \ge 1$. In particular, no power of X is the zero matrix.

Alternatively, one may square this X by hand and verify $X^2 = X$. It's then obvious that higher powers of X will also give X. Somewhat more rigorously, for $n \ge 2$,

$$X^{n} = X^{2} X^{n-2} = X X^{n-2} = X^{n-1} = \dots = X.$$