Math 308 M – Spring 2015
Proof Homework 1
Due Wednesday, May 6th, 2015

Name: ____________________

- **Answer one** of the following two questions. If you answer them both, your *better* answer will be ignored.
- Give rigorous proofs. Any skipped steps must be small enough that you could explain them to me in a few seconds. Your goal is to convince me you fully understand your argument and have not missed anything.
- You may use any theorem, proposition, etc. from lecture or the book, though when you do say at least "from the book" or "from lecture."
- For examples to model your proofs on, see the textbook, the proof examples document on the course web site, or any of the alternatives to the textbook linked from the course web site.
- You are welcome to talk to others (even outside the class) or work in groups on this assignment, though **write your final answers alone**. Keep in mind that this exercise is entirely for your benefit in becoming more comfortable with proofs.

1. Let X be a 2×2 matrix with trace 0.

   (a) Find constants α, β, γ with $\alpha \neq 0$ (which depend on the matrix X) so that

   $$\alpha X^2 + \beta X + \gamma I = 0.$$

   (Here I denotes the 2×2 identity matrix and 0 denotes the 2×2 zero matrix,

   $$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.)$$

   (b) Use the result of Proof Homework 1, Problem 2 and your answer from (a) to show that the columns of X are linearly dependent if and only if $X^2 = 0$.

2. Let X be an $n \times n$ matrix.
   (a) Suppose that $X^{100} = 0$. Show that the columns of X are linearly dependent.

   (b) Show that the converse of (a) does not hold. That is, find a matrix X whose columns are linearly dependent but where no power of X is the zero matrix.

   (Note: 1(b) can be interpreted as saying your counterexample cannot be a 2×2 trace 0 matrix.)