Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 56 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	19	
2	12	
3	14	
4	11	
Total:	56	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (1 point) A linear system is triangular if and only if its augmented matrix is triangular.

   ◯ True ✓ **False**

   (b) (5 points) Check all of the following properties of vector or matrix arithmetic which are *always true*. A, B, C are matrices, $\vec{u}$, $\vec{v}$ are column vectors, and s, t are scalars. Assume all dimensions are compatible.

   ◯ $AB = AC$ implies $B = C$ ✓ $A(BC) = (AB)C$ ✓ $(s+t)A = sA + tA$
   ◯ $(A+B)^2 = A^2 + 2AB + B^2$ ◯ $\mathbf{0} + I = \mathbf{0}I$ ✓ $A(s\vec{u} + t\vec{v}) = s(A\vec{u}) + t(A\vec{v})$
   ◯ If $ABC = CBA$, then $AC = CA$ ◯ $(sA)(sB) = s(AB)$ ✓ $A + A = 2A$

   (c) (4 points) Define **two** of the following three terms: one-to-one function; span; linearly independent. (Clearly specify which terms you are defining.)

   Solution:

   - One-to-one function: if $f\colon X \to Y$, then f is a one-to-one function if for all $x \in X$, there is at most one $y \in Y$ such that $f(x) = y$.
   - Span: if $V = \{\vec{v}_1, \ldots, \vec{v}_m\} \subset \mathbb{R}^n$, then the span of V is the set of all linear combinations of V,
   $$\operatorname{span} V = \{c_1\vec{v}_1 + \cdots + c_m\vec{v}_m : c_1, \ldots, c_m \in \mathbb{R}\}.$$
   - Linearly independent: if $V = \{\vec{v}_1, \ldots, \vec{v}_m\} \subset \mathbb{R}^n$, then V is linearly independent if the only solution to
   $$c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = \vec{0}$$
   is the trivial solution $c_1 = \cdots = c_m = 0$.

   (d) (5 points) Give an example of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and a linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ where (1) T is not one-to-one; (2) U is not onto; and (3) $(T \circ U)(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^m$.

   Solution: There are many examples. Using the underlying matrices, probably the simplest is
   $$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}.$$

   (e) (4 points) Give an example of a linear system whose augmented matrix is in echelon form with two pivots, but where the linear system has fewer than two solutions.

   Solution: There are many examples. Any triangular system with 2 variables and 2 equations qualifies. One of the simplest is
   $$\begin{aligned} x_1 + x_2 &= 0 \\ x_2 &= 0 \end{aligned}$$

2. Consider the matrix

$$M = \begin{pmatrix} 2 & 6 & 4 & 16 \\ 3 & 9 & 3 & 18 \\ 1 & 3 & 4 & 12 \end{pmatrix}.$$

(a) (6 points) Compute the reduced row echelon form (RREF) of M.

Solution: Row reduction gives

$$\begin{pmatrix} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(b) (4 points) Interpret M as the augmented matrix of a linear system.

- Write down the linear system.

Solution:

$$\begin{aligned} 2x_1 + 6x_2 + 4x_3 &= 16 \\ 3x_1 + 9x_2 + 3x_3 &= 18 \\ 4x_1 + 3x_2 + 4x_3 &= 12 \end{aligned}$$

- Write down the general solution of the linear system.

Solution:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 - 3s_1 \\ s_1 \\ 2 \end{pmatrix}$$

(c) (2 points) The rows of M:

◯ span $\mathbb{R}^4$ $\checkmark$ **do not span** $\mathbb{R}^4$

Briefly justify your answer.

Solution: There are only 3 rows, but at least 4 are needed to span $\mathbb{R}^4$. Alternatively, the second and fourth columns of the RREF of M have no pivots, so by a lemma from class the row space doesn't contain e_2 or e_4.

3. (a) (3 points) Find 4 different solutions to the matrix equation $A^2 = I_2$.

Solution: There are infinitely many examples, but some simple ones are

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

A nice infinite family is

$$\begin{pmatrix} n & n-1 \\ -(n+1) & -n \end{pmatrix}.$$

(b) (3 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

Compute A^{-1} if possible, or show it does not exist. (Do *not* use the "quick formula.")

Solution: We compute

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{bmatrix}$$

so that A^{-1} exists and is

$$A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}.$$

(c) (3 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2 \end{bmatrix}.$$

Is f linear? If so, prove it. If not, explain why not.

Solution: Not linear. For instance, $f(1,1) = (2,-1)$, but $f(2,2) = (4,0) \neq 2(2,-1)$. More simply, $f(0,0) = (0,-2) \neq (0,0)$.

(d) (5 points) Describe the following linear transformation geometrically–what does it do to an input to construct an output?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \cos -\pi/3 & \sin\pi/3 \\ \sin -\pi/3 & -\cos\pi/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Solution: It first rotates the input $\pi/3$ radians clockwise, scales the resulting vector in the x-direction by a factor of 2 and in the y-direction by a factor of 3, and then rotates the result back $\pi/3$ radians counterclockwise.

Alternatively, it scales the input in the $(\cos(\pi/3), \sin(\pi/3))$ direction by a factor of 2 and in the perpendicular $(-\sin(\pi/3), \cos(\pi/3))$ direction by a factor of 3.

4. This question is meant to be friendly–give it a chance!

   (a) (1 point) Write down your favorite matrix (or at least one you like well enough).

   Solution: Mine is perhaps the Vandermonde matrix,

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_1^2 & \cdots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \cdots & \alpha_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_n & \alpha_n^2 & \cdots & \alpha_n^{n-1} \end{pmatrix}.$$

   (b) (6 points) Suppose B is a 3×3 matrix and $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$. Prove that if we have vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ such that

$$B\vec{u}_1 = \vec{v}_1, \qquad B\vec{u}_2 = \vec{v}_2, \qquad B\vec{u}_3 = \vec{v}_3,$$

   then B is invertible.

   Solution: There are many possible arguments. Here's one. Since $\mathrm{col}(B)$ is the range of B, and each $\vec{v}_i$ is in the range of B, we have $V \subset \mathrm{col}(B)$. Since V spans $\mathbb{R}^3$, we have

$$\mathbb{R}^3 = \operatorname{span} V \subset \mathrm{col}(B),$$

   forcing $\mathrm{col}(B) = \mathbb{R}^3$. That is, the columns of B span $\mathbb{R}^3$, which is one of the conclusions of the Big Theorem. By the Big Theorem, B is invertible.

   (c) (4 points) Prove that if $A^2 = I$, then $T(\vec{x}) = A\vec{x}$ is one-to-one.

   Solution: There are many possible arguments. Here's one. Suppose $T(\vec{u}) = T(\vec{v})$, so $A\vec{u} = A\vec{v}$. Multiply on the left by A to get

$$\vec{u} = I\vec{u} = A^2\vec{u} = A^2\vec{v} = I\vec{v} = \vec{v}.$$

   Hence A is one-to-one directly from the definition.

   Alternatively, $A^2 = AA = I$. From a result in class, A is invertible with inverse $A^{-1} = A$, so T is invertible, and invertible is equivalent to being one-to-one and onto, so T is one-to-one.