Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 50 minutes for the exam.
- Check that you have a complete exam. There are 4 questions for a total of 56 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	19	
2	12	
3	14	
4	11	
Total:	56	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (1 point) A linear system is triangular if and only if its augmented matrix is triangular.

   ○ True ○ False

   (b) (5 points) Check all of the following properties of vector or matrix arithmetic which are *always true*. A, B, C are matrices, $\vec{u}$, $\vec{v}$ are column vectors, and s, t are scalars. Assume all dimensions are compatible.

   ○ $AB = AC$ implies $B = C$ ○ $A(BC) = (AB)C$ ○ $(s+t)A = sA + tA$
   ○ $(A+B)^2 = A^2 + 2AB + B^2$ ○ $\mathbf{0} + I = \mathbf{0}I$ ○ $A(s\vec{u} + t\vec{v}) = s(A\vec{u}) + t(A\vec{v})$
   ○ If $ABC = CBA$, then $AC = CA$ ○ $(sA)(sB) = s(AB)$ ○ $A + A = 2A$

   (c) (4 points) Define **two** of the following three terms: one-to-one function; span; linearly independent. (Clearly specify which terms you are defining.)

   (d) (5 points) Give an example of a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ and a linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ where (1) T is not one-to-one; (2) U is not onto; and (3) $(T \circ U)(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^m$.

   (e) (4 points) Give an example of a linear system whose augmented matrix is in echelon form with two pivots, but where the linear system has fewer than two solutions.

2. Consider the matrix

$$M = \begin{pmatrix} 2 & 6 & 4 & 16 \\ 3 & 9 & 3 & 18 \\ 1 & 3 & 4 & 12 \end{pmatrix}.$$

(a) (6 points) Compute the reduced row echelon form (RREF) of M.

(b) (4 points) Interpret M as the augmented matrix of a linear system.

- Write down the linear system.

- Write down the general solution of the linear system.

(c) (2 points) The rows of M:

◯ span $\mathbb{R}^4$ ◯ do not span $\mathbb{R}^4$

Briefly justify your answer.

3. (a) (3 points) Find 4 different solutions to the matrix equation $A^2 = I_2$.

(b) (3 points) Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}.$$

Compute A^{-1} if possible, or show it does not exist. (Do *not* use the "quick formula.")

(c) (3 points) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - 2 \end{bmatrix}.$$

Is f linear? If so, prove it. If not, explain why not.

(d) (5 points) Describe the following linear transformation geometrically–what does it do to an input to construct an output?

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \cos -\pi/3 & \sin \pi/3 \\ \sin -\pi/3 & -\cos \pi/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4. This question is meant to be friendly–give it a chance!

   (a) (1 point) Write down your favorite matrix (or at least one you like well enough).

   (b) (6 points) Suppose B is a 3×3 matrix and $V = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ spans $\mathbb{R}^3$. Prove that if we have vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ such that

   $$B\vec{u}_1 = \vec{v}_1, \qquad B\vec{u}_2 = \vec{v}_2, \qquad B\vec{u}_3 = \vec{v}_3,$$

   then B is invertible.

   (c) (4 points) Prove that if $A^2 = I$, then $T(\vec{x}) = A\vec{x}$ is one-to-one.