Math 308 M **Final** **Spring 2015**

Your Name

Student ID #

- Do not open this exam until you are told to begin. You will have 1 hour and 50 minutes for the exam.
- Check that you have a complete exam. There are 7 questions for a total of 110 points.
- You are allowed to have one handwritten note sheet. Only basic non-graphing scientific calculators are allowed, though you should not need one.
- Cheating will result in a zero and be reported to the Dean's Academic Conduct Committee.
- **Show all your work**. Unless explicitly stated otherwise in a particular question, if there is no work supporting your answer, you will not receive credit for the problem. If you need more space to answer a question, continue on the back of the page, and indicate that you have done so.

Question	Points	Score
1	21	
2	21	
3	18	
4	12	
5	12	
6	11	
7	15	
Total:	110	

1. Multiple choice and short answer. For these questions, you are **not required to show any work**.

   (a) (2 points) There are infinitely many one-dimensional subspaces of $\mathbb{R}^2$.

   ◯ True ◯ False

   (b) (2 points) If A is $n \times n$, then the reduced row echelon form of A is I_n.

   ◯ True ◯ False

   (c) (2 points) If A and B are $n \times n$ and $\det(AB) \neq 0$, then A and B are row equivalent.

   ◯ True ◯ False

   (d) (2 points) Let A be $m \times n$. Then $A^T A$ is symmetric if and only if $m = n$. (Recall that X is symmetric if $X = X^T$.)

   ◯ True ◯ False

   (e) (2 points) A nonsingular matrix can have 0 as an eigenvalue.

   ◯ True ◯ False

   (f) (2 points) If $S \subset \mathbb{R}^4$ is a subspace of dimension 2, then every $\mathbf{x} \in \mathbb{R}^4$ is in either S or $S^\perp$.

   ◯ True ◯ False

   (g) (2 points) Let A be an $n \times n$ matrix with (distinct) eigenvalues $\lambda_1, \ldots, \lambda_k$ and eigenspaces $S_1, \ldots, S_k$. Then $\dim S_1 + \cdots + \dim S_k \leq n$.

   ◯ True ◯ False

   (h) (2 points) A subspace $S \neq \{\mathbf{0}\}$ can have a finite number of vectors.

   ◯ True ◯ False

   (i) (3 points) Give the definition of "subspace."

   (j) (2 points) Give the definition of "basis."

2. Provide examples meeting the given requirements. Unlike on the midterms, you **must justify your answers** on this question.

   (a) (4 points) Give a 3×3 matrix which has π as an eigenvalue where the π-eigenspace has dimension 3.

   (b) (4 points) Find A and B where $\det(A + B) \neq \det(A) + \det(B)$.

   (c) (5 points) Find a 3×3 matrix A where $A^3 = 0$ but $A^2 \neq 0$. (Hint: triangular matrices.)

(d) (4 points) Find a matrix whose characteristic polynomial is $(1-\lambda)(2-\lambda)^2(3-\lambda)^3$.

(e) (4 points) Give an example of a one-to-one linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ together with another linear transformation $U\colon \mathbb{R}^m \to \mathbb{R}^n$ where $U \circ T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the identity, i.e. $U(T(\mathbf{x})) = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

3. Prove each of the following statements. Hint: each part is independent of the others unless stated otherwise.

   (a) (2 points) Let A be a square matrix. Show that $A^3 - I = (A - I)(A^2 + A + I)$.

   (b) (2 points) If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are (column) vectors, show that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

   (c) (5 points) Show that if λ is an eigenvalue of $A^T A$, then $\lambda \geq 0$. (Hint: if $\mathbf{v}$ is an eigenvector of $A^T A$ with eigenvalue λ, show that $\lambda |\mathbf{v}|^2 = (A^T A\mathbf{v}) \cdot \mathbf{v} = (A\mathbf{v}) \cdot (A\mathbf{v}) \geq 0$ using (b) twice.)

(d) (2 points) Let A and P be $n \times n$ matrices with P invertible and let λ be a scalar. Show that $PAP^{-1} - \lambda I = P(A - \lambda I)P^{-1}$

(e) (3 points) Let A and P be $n \times n$ matrices with P invertible. Use (d) to show that A and PAP^{-1} have the same characteristic polynomial.

(f) (4 points) Suppose X and Y are square matrices which commute, meaning $XY = YX$. Show that if $\mathbf{u} \in \text{null}(X)$, then $Y\mathbf{u} \in \text{null}(X)$.

4. Let

$$A := \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (a) (4 points) Compute $\det(A)$. Is A invertible?

   (b) (4 points) Let L be the lower triangular matrix above and let U be the upper triangular matrix, so $A = LU$. Show that $\text{null}(A) = \text{null}(U)$. (Hint: L is invertible.)

   (c) (4 points) Show that U is an echelon form of A. Find a basis for $\text{null}(A)$.

5. (a) (5 points) Let A be a 2×2 matrix where

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix}.$$

What is A?

(b) (3 points) Find bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^2$ such that the matrix A from (a) is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$. (If you did not solve (a), you may replace A with your own 2×2 matrix.)

(c) (4 points) Find the change of basis matrix from the basis

$$\left\{ \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

to the basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

6. Let

$$A = \begin{bmatrix} 0 & 0 & -2 & -1 \\ 1 & 1 & 6 & 5 \\ 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (4 points) Compute the characteristic polynomial of A directly. (Hint: the eigenvalues of A are 1 and 2.)

(b) (5 points) Compute a basis for the eigenspace of 1.

(c) (2 points) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$. Compute $A^{100}\mathbf{x}$.

7. The following is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 5 \end{bmatrix} \right\}.$$

   (a) (5 points) Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

   (b) (3 points) Find an ortho**normal** basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathbb{R}^3$ where $\mathbf{u}_1 \cdot \mathbf{v}_1 = |\mathbf{u}_1||\mathbf{v}_1|$ where $\mathbf{v}_1$ is as in (a).

(c) (3 points) Let S be $\operatorname{span}\{\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_5\} \subset \mathbb{R}^5$ and let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}$. Compute $\operatorname{proj}_S \mathbf{y}$.

(d) (4 points) Find a least squares solution $\widehat{\mathbf{x}}$ to the system $A\mathbf{x} = \mathbf{y}$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 5 \end{bmatrix}.$$

Are there any other least squares solutions?