1. (10 points) Solve the following initial value problem:

$$\frac{dy}{dx} - \ln(x) \cdot y^2 = 0, \qquad y(1) = \frac{1}{2}$$

Your answer should be a function $y(x)$ with no undetermined constants in it.

Separable first-order DE.
Solving for each variable on its own side:

$\frac{1}{y^2} dy = \ln(x)\, dx$. So $\int \frac{1}{y^2} dy = \int \ln(x)\, dx$

LHS: $\int \frac{1}{y^2} dy = -\frac{1}{y}$.

RHS: Do u-sub. $u = \ln(x)$ or $x = e^u$
$du = \frac{1}{x} dx$ $\quad dx = e^u du$

So $\int \ln(x) dx = \int u e^u du$

Now do IBP: $f = u$ $\quad g = e^u$
$df = du$ $\quad dg = e^u du$

So $\int u e^u du = ue^u - \int e^u du$
$= ue^u - e^u$
$= (u-1)e^u$
$= (\ln(x) - 1)e^{\ln(x)}$
$= x(\ln(x) - 1)$.

So $-\frac{1}{y} = x(\ln(x) - 1) + C$

$\Rightarrow y = \frac{1}{D + x - x\ln(x)}$, $D = -C$.

Apply IC: $y(1) = \frac{1}{2} \Rightarrow \frac{1}{2} = \frac{1}{D + 1 - 1 \cdot 0}$

$\Rightarrow D = 1$

So solution is $\boxed{y = \frac{1}{1 + x - x\ln(x)}}$

2. (10 points) Consider the non-homogeneous differential equation

$$y'' + 4y' - 21y = g(t),$$

for some nonzero forcing function $g(t)$. For each of the following possibilities for $g(t)$, write down the form that the particular solution $Y(t)$ to the DE would take. Your answer should be in the form $Y = f(t)$, where $f$ includes undetermined coefficients ($A, B, C$ etc.). For example, if you thought the particular solution was a general linear function in $t$, you would write $Y = At + B$. **You don't need to compute the actual values of these coefficients**.

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

(a) $g(t) = \cos(t)$

CE for the homogeneous DE is $r^2 + 4r - 21 = 0$
$\Rightarrow (r+7)(r-3) = 0$
So $y_1 = e^{-7t}$ & $y_2 = e^{3t}$ are solutions to the homogeneous DE

"cos & sin functions hunt in packs"

$\boxed{Y(t) = A\cos(t) + B\sin(t)}$

(b) $g(t) = e^t - 1$

- $e^t$ isn't a solution to homog. equation $\Rightarrow$ guess $Y_1(t) = Ae^t$
- $-1$ is a const $\Rightarrow$ guess constant solution $Y_2(t) = B$

$\Rightarrow \boxed{Y(t) = Ae^t + B}$

(c) $g(t) = t^2 - t$

$t^2 - t$ is a quadratic polynomial $\Rightarrow$ guess general degree-2 polynomial

$\Rightarrow \boxed{Y(t) = At^2 + Bt + C}$

(d) $g(t) = e^{3t} + e^{-7t}$

Both $e^{3t}$ & $e^{-7t}$ are solutions to the homogeneous DE $\Rightarrow$ bump up both guesses by a power of $t$.

$\Rightarrow \boxed{Y(t) = Ate^{3t} + Bte^{-7t}}$

(e) $g(t) = e^{-2t}\sin 5t$

"cos & sin functions hunt in packs" Derivatives of $e^{2t}\sin(5t)$ will include $e^{2t}\sin(5t)$ & $e^{2t}\cos(5t)$ terms. Likewise with derivatives of $e^{2t}\cos(5t)$.

$\Rightarrow \boxed{Y(t) = e^{-2t}(A\sin(5t) + B\cos(5t))}$

3. (10 total points)

   (a) (3 points) Compute the Laplace transform of

$$f(t) = \sin^2(t)$$

You may quote any formula listed in the table of Laplace transforms at the back of the exam. [Hint: $\sin^2(t) = \frac{1-\cos(2t)}{2}$ ].

$$\mathcal{L}[\sin^2(t)] = \mathcal{L}\left[\frac{1-\cos(2t)}{2}\right] = \mathcal{L}[\tfrac{1}{2}] - \mathcal{L}[\tfrac{1}{2}\cos(2t)]$$
$$= \tfrac{1}{2}\mathcal{L}[1] - \tfrac{1}{2}\mathcal{L}[\cos(2t)]$$
$$= \frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+4}.$$

we can combine the terms to get $\boxed{\mathcal{L}[\sin^2(t)] = \frac{2}{s(s^2+4)}}$

   (b) (7 points) Use your answer above to compute the Laplace transform of the solution to the initial value problem

$$y'' + y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 2$$

where

$$g(t) = \begin{cases} \sin^2(t), & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

Your answer should be a function $\Phi(s)$ with no undetermined constants in it. **You do not need to find the solution to the IVP to answer this question.**

Let $\phi(t)$ solve the IVP, $\Phi(s) = \mathcal{L}[\phi(t)]$.

Then $\phi'' + \phi' + \phi = g(t)$, $\phi(0) = 0$, $\phi'(0) = 2$.

$\Rightarrow \mathcal{L}[\phi''] + \mathcal{L}[\phi'] + \mathcal{L}[\phi] = \mathcal{L}[g(t)]$

$\Rightarrow s^2\Phi - s\cdot\cancel{\phi(0)} - \underset{2}{\phi'(0)} + s\Phi - \cancel{\phi(0)} + \Phi = \mathcal{L}[g(t)]$

$(s^2+s+1)\Phi - 2 = \mathcal{L}[g(t)]$

Now $g(t) = \sin^2(t) - u_\pi(t)\cdot\sin^2(t)$

$= \sin^2(t) - u_\pi(t)\cdot\sin^2(t-\pi)$, since $\sin^2(t)$ is periodic with period $\pi$

$\Rightarrow \mathcal{L}[g(t)] = \mathcal{L}[\sin^2(t)] - e^{-\pi s}\cdot\mathcal{L}[\sin^2(t)]$

$= (1-e^{-\pi s})\mathcal{L}[\sin^2(t)]$

$= \frac{(1-e^{-\pi s})\cdot 2}{s(s^2+4)}$

Here $(s^2+s+1)\Phi - 2 = \frac{(1-e^{-\pi s})\cdot 2}{s(s^2+4)}$

Solving for $\Phi$:

$$\boxed{\Phi(s) = \frac{2}{s^2+s+1} + \frac{2(1-e^{-\pi s})}{s(s^2+4)(s^2+s+1)}}$$

4. (10 total points) Consider the following differential equation:

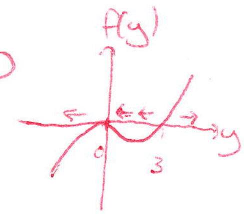
$$\frac{dy}{dx} = \frac{y^3 - 3y^2}{y^2+1} = f(y).$$

(a) (4 points) Find all equilibrium solutions to the DE, and classify them according to their stability.

Note: the denominator is $>0$ for any $y$, so it doesn't change the values or stability of any equilibrium points

So $y' = \frac{y^3 - 3y^2}{y^2+1}$ has a ~~stable~~ equilibrium point wherever $y^3 - 3y^2 = 0$

And $y^3 - 3y^2 = y^2(y-3) = 0 \Rightarrow y = 0$ or $y = 3$.




Now $f(y)$ is +ve to the right of $y=3$, −ve to the left $\Rightarrow$ unstable.

And $f(y)$ is −ve to both sides of $y=0$ $\Rightarrow$ Semistable.

$\Rightarrow$ $\boxed{y=0 \text{ is a semistable eq. solution}}$, $\boxed{y=3 \text{ is unstable}}$. No other eq. solutions.

(b) (4 points) Let $y = \phi(t)$ be the solution to the above DE subject to the initial condition $y(0) = 1$. Use Euler's Method with a step size of $h = 0.5$ to find an approximate value of the solution at $t = 1$. You may use decimals in this part of the question (although you don't need to); if you do be sure to maintain at least four digits of precision.

Need 2 steps to get to $x=1$ using step size of $h = \frac{1}{2}$.

Start: $x_0 = 0$, $y_0 = 1$

$x_1 = \frac{1}{2}$, $y_1 = y_0 + h \cdot f(\cancel{x_0}, y_0)$ — no $x$ dependence

$= 1 + \frac{1}{2}\cdot\left(\frac{1^3 - 3\cdot 1^2}{1^2+1}\right)$

$= 1 + \frac{1}{2}\left(\frac{-2}{2}\right)$

$= \frac{1}{2}$

And $x_2 = 1$, $y_2 = y_1 + h f(y_1)$ — last step!

$= \frac{1}{2} + \frac{1}{2}\left(\frac{(\frac{1}{2})^3 - 3\cdot(\frac{1}{2})^2}{(\frac{1}{2})^2+1}\right) \times \frac{8}{8}$

$= \frac{1}{2} + \frac{1}{2}\left(\frac{1-6}{2+8}\right)$

$= \frac{1}{2} + \frac{1}{2}\cdot\left(\frac{-5}{10}\right)$

$= \frac{1}{4}$. Stop here.

So $\boxed{y_2 = \frac{1}{4} \text{ is an approx value of the true solution @ } t=1}$

(c) (2 points) Let $y = \phi(t)$ be the solution mentioned in the previous part of the question. What is $\lim_{t\to\infty}\phi(t)$? Justify your answer.

$\lim_{t\to\infty}\phi(t) = 0$. This is because $y=0$ is a semistable solution, with solutions $> y=0$ and $< y=3$ decreasing to 0 (and solutions $< y=0$ decreasing to $-\infty$).

Since $\phi(0) = 1$, it sits between 0 & 3

$\Rightarrow$ it must $\to 0$ as $t \to \infty$.

5. (10 total points) An applied mathematician is investigating the motion of a particular object, and establishes that the function describing its motion $y(t)$ obeys the differential equation

$$y'' + by' + cy = 0$$

where $a$ and $b$ are constants. The mathematician doesn't initially know the values of $b$ and $c$, but can show the following two facts:

- The function $y_1(t) = e^{-4t}$ is a solution to the differential equation.
- The Wronskian of the system is $W(t) = e^{-8t}$.

(a) (7 points) Using the above two facts, find a second function $y_2(t)$, linearly independent from the first, that satisfies the differential equation. Your answer should be a function in $t$ **with no undetermined coefficients in it.**

We have $e^{-8t} = W(t) = y_1 y_2' - y_1' y_2$
$= e^{-4t} y_2' - (-4) e^{-4t} y_2$

$\Rightarrow e^{-4t} y_2' + 4e^{-4t} y_2 = e^{-8t}$

$\Rightarrow y_2' + 4y_2 = e^{-4t}$

This is a first-order linear DE, which we can solve!

$\mu(t) = e^{\int 4\,dt} = e^{4t}$

So $y_2(t) = e^{-4t}\left(\int e^{4t} \cdot e^{-4t}\,dt + C\right)$
$= Ce^{-4t} + e^{-4t}\int 1\,dt$
$= Ce^{-4t} + te^{-4t}$

Choosing $C = 0$ (as $e^{-4t}$ is already covered by $y_1(t)$)

$\Rightarrow y_2(t) = te^{-4t}$.

That is, $\boxed{y_2(t) = te^{-4t}}$ also solves the DE.

(b) (3 points) Given that the functions $y_1(t)$ and $y_2(t)$ both solve the DE, what are the constants $b$ and $c$?

$y_1(t) = e^{-4t}$, $y_2(t) = te^{-4t}$.

So $y = c_1 e^{-4t} + c_2 te^{-4t} = (c_1 + c_2 t)e^{-4t}$ is a general solution to the DE

$\Rightarrow$ CE equation must be $(r+4)^2 = 0$, as we have a double root. $\Rightarrow r^2 + 8r + 16 = 0$

$\Rightarrow \boxed{b = 8, \quad c = 16}$

6. (10 total points) You've seen in class that inverse Laplace transforms obey some convenient rules which makes computing them a lot easier. For example, we have that $\mathscr{L}^{-1}[F(s)+G(s)] = \mathscr{L}^{-1}[F(s)] + \mathscr{L}^{-1}[G(s)]$, and $\mathscr{L}^{-1}[c \cdot F(s)] = c \cdot \mathscr{L}^{-1}[F(s)]$ for $c$ a constant.

   Below you are given two **FALSE** rules about how inverse Laplace transforms work. Your task is to provide specific examples of functions that prove these rules wrong.

   (a) (5 points) False rule # 1:

   $$\mathscr{L}^{-1}[F(s) \cdot G(s)] = \mathscr{L}^{-1}[F(s)] \cdot \mathscr{L}^{-1}[G(s)]$$

   Find two functions $F(s)$ and $G(s)$ for which this equation isn't true, and demonstrate that this is the case by stating the relevant inverse Laplace transforms. You may quote any formula given in the Laplace transform formula sheet at the back of the exam paper.

Many functions $F(s)$ & $G(s)$ will suffice.

We use $F(s) = \frac{1}{s}$ & $G(s) = \frac{1}{s^2}$

Then $\mathcal{L}^{-1}[F(s) \cdot G(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2} \cdot \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] = \frac{1}{2}t^2.$

while $\mathcal{L}^{-1}[F(s)] \cdot \mathcal{L}^{-1}[G(s)] = \mathcal{L}\left[\frac{1}{s}\right] \cdot \mathcal{L}\left[\frac{1}{s^2}\right] = 1 \cdot t = t.$

So $\mathcal{L}^{-1}[F(s)G(s)] = \frac{1}{2}t^2 \neq t = \mathcal{L}^{-1}[F(s)]\mathcal{L}^{-1}[G(s)].$

So clearly the rule is false.

   (b) (5 points) False rule # 2:

   $$\mathscr{L}^{-1}\left[\frac{d}{ds}F(s)\right] = \frac{d}{dt}\mathscr{L}^{-1}[F(s)]$$

   Find a function $F(s)$ for which this equation isn't true, and demonstrate that this is the case by stating the relevant inverse Laplace transforms. You may quote any formula given in the Laplace transform formula sheet at the back of the exam paper.

Again, many $F(s)$ will suffice. We use $F(s) = \frac{1}{s}$

Then $\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = \mathcal{L}^{-1}\left[\frac{d}{ds}\frac{1}{s}\right] = \mathcal{L}^{-1}\left[-\frac{1}{s^2}\right] = -t.$

while $\frac{d}{dt}\mathcal{L}^{-1}[F(s)] = \frac{d}{dt}\mathcal{L}\left[\frac{1}{s}\right] = \frac{d}{dt}(1) = 0.$

So $\mathcal{L}^{-1}\left[\frac{d}{ds}F(s)\right] = -t \neq 0 = \frac{d}{dt}\mathcal{L}^{-1}[F(s)].$

So clearly this rule is false too.

7. (10 total points) A 5 kg block is placed on a flat surface and attached to a long horizontal spring. When the block is pulled 0.2 meters to the right of its equilibrium position, the spring exerts a force of 2.5 Newtons to the left on the block. Furthermore the surface imparts a frictional force on the block proportional to its velocity, such that when the block is traveling at 1 $\text{ms}^{-1}$ the retarding force is 9 Newtons. No other forces act on the block.

   At time $t = 0$ the block is fired from its equilibrium position with a velocity of 1 $\text{ms}^{-1}$ to the right.

   (a) (2 points) Write down an initial value problem describing the position of the block as a function of time. $my'' + \gamma y' + ky = g(t)$.

   For us : $m = 5$, $\gamma = 9$, $k = \frac{2.5}{0.2} = \frac{25}{2}$, and $g(t) = 0$.

   And $y(0) = 0$, $y'(0) = 1$

   $\Rightarrow \boxed{5y'' + 9y' + \frac{25}{2}y = 0, \quad y(0) = 0, \quad y'(0) = 1}$

   (b) (5 points) Solve this initial value problem to find a formula for the position of the block at time $t$ for $t \geq 0$.

   CE: $5r^2 + 9r + \frac{25}{2} = 0$

   So $r = \dfrac{-9 \pm \sqrt{9^2 - 4 \cdot 5 \cdot \frac{25}{2}}}{2 \cdot 5}$

   $= \frac{-9}{10} \pm \frac{1}{10}\sqrt{81 - 250}$

   $= \frac{-9}{10} \pm \frac{1}{10}\sqrt{-169}$

   $= \frac{-9}{10} \pm \frac{13}{10} i$

   So general solution is

   $y = e^{-\frac{9}{10}t}\left(C_1 \cos\left(\frac{13}{10}t\right) + C_2 \sin\left(\frac{13}{10}t\right)\right)$

   ICs: $y(0) = 0 \Rightarrow C_1 = 0 \Rightarrow y = C_2 e^{-\frac{9}{10}t} \sin\left(\frac{13}{10}t\right)$

   And $y' = -\frac{9}{10}C_2 e^{-\frac{9}{10}t}\sin\left(\frac{13}{10}\right) + \frac{13}{10}C_2 e^{-\frac{9}{10}t}\cos\left(\frac{13}{10}t\right)$

   So $y'(0) = 1 \Rightarrow 1 = \frac{13}{10}C_2 \Rightarrow C_2 = \frac{10}{13}$

   So $\boxed{y(t) = \frac{10}{13} e^{-\frac{9}{10}t} \sin\left(\frac{13}{10}t\right)}$

   (c) (3 points) When will the block first cross back over its equilibrium position?

   Block crosses equilibrium point whenever $y(t) = 0$

   And $y(t) = 0 \Rightarrow \frac{10}{13}e^{-\frac{9}{10}t}\sin\left(\frac{13}{10}t\right) = 0 \Rightarrow \sin\left(\frac{13}{10}t\right) = 0$ as $\frac{10}{13}e^{-\frac{9}{10}t}$ is never zero. So $y(t) = 0 \Rightarrow \frac{13}{10}t = n \cdot \pi$ for any integer $n$

   We want first time after $t = 0$, so choose $n = 1$

   $\Rightarrow \frac{13}{10}t = \pi \Rightarrow \boxed{t = \frac{10}{13}\pi}$ is when the block first crosses back over the equilibrium point.

8. (10 points + 4 bonus points) A small island in the Atlantic is having a problem with its invasive rat population. The residents of the island worriedly note that the rat population is growing at a rate proportional to its own size, increasing in size by a factor of $e$ every 10 months (where $e = 2.71828\ldots$).

   To counter this, the residents bring in a shipment of cats, who upon arrival catch and kill rats at an initial rate of 1000 a month. However, the cats grow more skilled in their rat-catching efforts as time goes on, and as such the number of rats they catch increases by 100 every month.

   (a) (10 points) The cats arrive at the beginning of the year, when there are 5000 rats on the island. Establish and solve an initial value problem to find the number of rats on the island after the cats arrive as a function of time $t$.

Let $y(t)$ be the size of the rat population at time $t$, $t$ in months, $t=0$ when the cats arrive.

Without cats, the rats grow at a rate proportional to the population size:

$$\frac{dy}{dt} = \alpha y \quad \text{for some } \alpha$$

$$\Rightarrow \frac{1}{y}dy = \alpha\, dt$$

$$\Rightarrow \ln(y) = \alpha t + C$$

$$\Rightarrow y = Ae^{\alpha t}$$

And the population size grows by a factor of $e$ every 10 months means $\frac{y(10)}{y(0)} = e^1$. $\frac{y(10)}{y(0)} = \frac{Ae^{10\alpha}}{Ae^0} = e^{10\alpha}$

So $10\alpha = 1 \Rightarrow \alpha = \frac{1}{10}$

However, cats eat rats at a rate of $1000 + 100t$

So after cats arrive, the rat population obeys the DE

$$\frac{dy}{dt} = \frac{y}{10} - (1000 + 100t), \text{ with } y(0) = 5000.$$

This is a 1st-order linear DE: Standard form is $\frac{dy}{dt} - \frac{1}{10}y = -1000 - 100t$

$\mu(t) = e^{\int -\frac{1}{10}dt} = e^{-\frac{t}{10}}$. So $y(t) = e^{\frac{t}{10}}\left(\int (-1000-100t)e^{-\frac{t}{10}}dt + C\right)$

$= Ce^{\frac{t}{10}} - 100\left(\int (10+t)e^{-\frac{t}{10}}dt\right)\cdot e^{\frac{t}{10}}$

To compute ~~$\int(-1000 - 100t$~~ $\int (10+t)e^{-\frac{t}{10}}dt$, use integration by parts:

PTO

[Question continued on the next page]

$$u = 10+t \qquad v = -10e^{-\frac{t}{10}}$$

$$du = dt \qquad dv = e^{-\frac{t}{10}}\,dt$$

So $\displaystyle\int (10+t)e^{-\frac{t}{10}}\,dt = -10(10+t)e^{-\frac{t}{10}} + 10\int e^{-\frac{t}{10}}\,dt$

$$= -10(10+t)e^{-\frac{t}{10}} - 100\,e^{-\frac{t}{10}}$$

$$= (-200-10t)e^{-\frac{t}{10}}$$

Thus $\displaystyle y(t) = Ce^{\frac{t}{10}} - 100\,e^{\frac{t}{10}}\left(\int (t+10)e^{-\frac{t}{10}}\,dt\right)$

$$= Ce^{\frac{t}{10}} - 100\,\cancel{e^{\frac{t}{10}}}\left(-200-10t\right)\cancel{e^{-\frac{t}{10}}}$$

$$= 20000 + 1000t + Ce^{\frac{t}{10}}$$

Now apply ICs: $y(0) = 5000$

$\Rightarrow \quad 5000 = 20000 + 1000\cdot 0 + Ce^{0}$

$\Rightarrow \quad C = -15000$

So $$\boxed{y(t) = 1000\left(20+t-15e^{\frac{t}{10}}\right)}$$

(b) (Bonus: 4 points) Compute or estimate how long it would take for the cats to completely eliminate the island's rat population.

The cats eliminate the rat population when $y(t) = 0$.

$\Rightarrow \quad 1000\left(20 + t - 15e^{\frac{t}{10}}\right) = 0$

$\Rightarrow \quad 20 + t - 15e^{\frac{t}{10}} = 0$

or the $t$ when $20 + t = 15e^{\frac{t}{15}}$

This is not an equation that can be solved explicitly, so we'll have to use some sort of approximation. Now ~~&~~ by sketching $15e^{\frac{t}{10}}$ vs. $20+t$ roughly, we can see that the solution is a $t$-value that isn't too big i.e. somewhere between $t = 1$ & $10$

$\Rightarrow \frac{t}{10} < 1$ so we can use a Taylor series approximation for $e^{\frac{t}{10}}$

Using $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ to get $e^{\frac{t}{10}} = 1 + \frac{t}{10} + \frac{t^2}{2 \cdot 10^2} + \dots$

we use the approximation therefore that $e^{\frac{t}{10}} \simeq 1 + \frac{t}{10} + \frac{t^2}{200}$ for $t$ not too big.

$\Rightarrow$ We solve $20 + t = 15\left(1 + \frac{t}{10} + \frac{t^2}{200}\right)$

$\Rightarrow \quad 5 + \cancel{t} = \frac{3}{2}t + \frac{3}{40}t^2$

$\Rightarrow \quad \frac{3}{40}t^2 + \frac{1}{2}t - 5 = 0$

$\Rightarrow \quad 3t^2 + 20t - 200 = 0$

$$\Rightarrow \quad t = \frac{-20 \pm \sqrt{20^2 - 4 \cdot 3 \cdot (-200)}}{2 \cdot 3}$$

$$= \frac{-10}{3} \pm \frac{1}{6}\sqrt{400 + 2400}$$

$$= \frac{-10}{3} \pm \frac{\sqrt{2800}}{6} = -\frac{10}{3} \pm \frac{10}{3}\sqrt{7}.$$

so $t = \frac{10}{3}(-1 \pm \sqrt{7})$

Now $\frac{10}{3}(-1-\sqrt{7}) < 0 \Rightarrow$ doesn't make sense, so

choose $\boxed{t = \frac{10}{3}(-1+\sqrt{7}) \simeq 5.4858 \text{ months}}$

This is in fact close to the true numerical solution of $t = 5.18$ months so our approximation is pretty good.