Math 307E, Autumn 2014 **Final Exam** **University of Washington**

Your Name

Student ID #

- **DON'T PANIC!** There are more questions in this exam than in the midterms, but you have more than twice the time to solve them. If you get stuck, take a deep breath and move on to something else. Return if you have time at the end.
- Cellphones off please!
- You are allowed one two-sided handwritten notesheet for this midterm.
- You may use a scientific calculator, although this exam has been written so that you don't need to. Graphing calculators and all other course-related materials may not be used.
- In order to receive full credit, you must **show your working** unless explicitly stated otherwise by the question. You may quote and use any formula you have seen in class to save time, but be sure to indicate with a word or two when you are doing so.
- **There is a table of Laplace transforms and rules at the back of this exam**. You may quote and use any of the formulas and rules in the table as is without having to derive them from scratch.
- Give your answers in exact form (for example $\pi/3$ or $e^{-5\sqrt{3}}$) unless explicitly stated otherwise by the question. Simplify your answers if possible.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 10 pages, plus this cover sheet. Please make sure that your exam is complete.
- You have 110 minutes to complete the exam.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10 points) Solve the following initial value problem:

$$\frac{dy}{dx} - \ln(x) \cdot y^2 = 0, \qquad y(1) = \frac{1}{2}$$

Your answer should be a function $y(x)$ with no undetermined constants in it.

2. (10 points) Consider the non-homogeneous differential equation

$$y'' + 4y' - 21y = g(t),$$

for some nonzero forcing function $g(t)$. For each of the following possibilities for $g(t)$, write down the form that the particular solution $Y(t)$ to the DE would take. Your answer should be in the form $Y = f(t)$, where f includes undetermined coefficients (A, B, C etc.). For example, if you thought the particular solution was a general linear function in t, you would write $Y = At + B$. **You don't need to compute the actual values of these coefficients**.

Each part is worth 2 points. You don't need to show your working to get full credit for this question.

(a) $g(t) = \cos(t)$

(b) $g(t) = e^t - 1$

(c) $g(t) = t^2 - t$

(d) $g(t) = e^{3t} + e^{-7t}$

(e) $g(t) = e^{-2t} \sin 5t$

3. (10 total points)

   (a) (3 points) Compute the Laplace transform of

   $$f(t) = \sin^2(t)$$

   You may quote any formula listed in the table of Laplace transforms at the back of the exam. [Hint: $\sin^2(t) = \frac{1-\cos(2t)}{2}$].

   (b) (7 points) Use your answer above to compute the Laplace transform of the solution to the initial value problem

   $$y'' + y' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 2$$

   where

   $$g(t) = \begin{cases} \sin^2(t), & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

   Your answer should be a function $\Phi(s)$ with no undetermined constants in it. **You do not need to find the solution to the IVP to answer this question.**

4. (10 total points) Consider the following differential equation:

$$\frac{dy}{dx} = \frac{y^3 - 3y^2}{y^2 + 1}$$

(a) (4 points) Find all equilibrium solutions to the DE, and classify them according to their stability.

(b) (4 points) Let $y = \phi(t)$ be the solution to the above DE subject to the initial condition $y(0) = 1$. Use Euler's Method with a step size of $h = 0.5$ to find an approximate value of the solution at $t = 1$. You may use decimals in this part of the question (although you don't need to); if you do be sure to maintain at least four digits of precision.

(c) (2 points) Let $y = \phi(t)$ be the solution mentioned in the previous part of the question. What is $\lim_{t\to\infty} \phi(t)$? Justify your answer.

5. (10 total points) An applied mathematician is investigating the motion of a particular object, and establishes that the function describing its motion $y(t)$ obeys the differential equation

$$y'' + by' + cy = 0$$

where a and b are constants. The mathematician doesn't initially know the values of b and c, but can show the following two facts:

- The function $y_1(t) = e^{-4t}$ is a solution to the differential equation.
- The Wronskian of the system is $W(t) = e^{-8t}$.

(a) (7 points) Using the above two facts, find a second function $y_2(t)$, linearly independent from the first, that satisfies the differential equation. Your answer should be a function in t **with no undetermined coefficients in it.**

(b) (3 points) Given that the functions $y_1(t)$ and $y_2(t)$ both solve the DE, what are the constants b and c?

6. (10 total points) You've seen in class that inverse Laplace transforms obey some convenient rules which makes computing them a lot easier. For example, we have that $\mathscr{L}^{-1}[F(s)+G(s)] = \mathscr{L}^{-1}[F(s)] + \mathscr{L}^{-1}[G(s)]$, and $\mathscr{L}^{-1}[c \cdot F(s)] = c \cdot \mathscr{L}^{-1}[F(s)]$ for c a constant.

   Below you are given two **FALSE** rules about how inverse Laplace transforms work. Your task is to provide specific examples of functions that prove these rules wrong.

   (a) (5 points) False rule # 1:

   $$\mathscr{L}^{-1}[F(s) \cdot G(s)] = \mathscr{L}^{-1}[F(s)] \cdot \mathscr{L}^{-1}[G(s)]$$

   Find two functions $F(s)$ and $G(s)$ for which this equation isn't true, and demonstrate that this is the case by stating the relevant inverse Laplace transforms. You may quote any formula given in the Laplace transform formula sheet at the back of the exam paper.

   (b) (5 points) False rule # 2:

   $$\mathscr{L}^{-1}\left[\frac{d}{ds}F(s)\right] = \frac{d}{dt}\mathscr{L}^{-1}[F(s)]$$

   Find a function $F(s)$ for which this equation isn't true, and demonstrate that this is the case by stating the relevant inverse Laplace transforms. You may quote any formula given in the Laplace transform formula sheet at the back of the exam paper.

7. (10 total points) A 5 kg block is placed on a flat surface and attached to a long horizontal spring. When the block is pulled 0.2 meters to the right of its equilibrium position, the spring exerts a force of 2.5 Newtons to the left on the block. Furthermore the surface imparts a frictional force on the block proportional to its velocity, such that when the block is traveling at 1 ms^{-1} the retarding force is 9 Newtons. No other forces act on the block.

   At time $t = 0$ the block is fired from its equilibrium position with a velocity of 1 ms^{-1} to the right.

   (a) (2 points) Write down an initial value problem describing the position of the block as a function of time.

   (b) (5 points) Solve this initial value problem to find a formula for the position of the block at time t for $t \geq 0$.

   (c) (3 points) When will the block first cross back over its equilibrium position?

8. (10 points + 4 bonus points) A small island in the Atlantic is having a problem with its invasive rat population. The residents of the island worriedly note that the rat population is growing at a rate proportional to its own size, increasing in size by a factor of e every 10 months (where $e = 2.71828\ldots$).

   To counter this, the residents bring in a shipment of cats, who upon arrival catch and kill rats at an initial rate of 1000 a month. However, the cats grow more skilled in their rat-catching efforts as time goes on, and as such the number of rats they catch increases by 100 every month.

   (a) (10 points) The cats arrive at the beginning of the year, when there are 5000 rats on the island. Establish and solve an initial value problem to find the number of rats on the island after the cats arrive as a function of time t.

[Question continued on the next page]

(b) (Bonus: 4 points) Compute or estimate how long it would take for the cats to completely eliminate the island's rat population.

Table of Laplace Transforms

In this table, n always represents a positive integer, and a and c are real constants.

$f(t) = \mathscr{L}^{-1}[F(s)]$	$F(s) = \mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$	
1	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^n e^{ct}$, n a positive integer	$\frac{n!}{(s-c)^{n+1}}$	$s > c$
t^a, $a > -1$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \lvert a \rvert$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \lvert a \rvert$
$e^{ct}\cos(at)$	$\frac{s-c}{(s-c)^2+a^2}$	$s > c$
$e^{ct}\sin(at)$	$\frac{a}{(s-c)^2+a^2}$	$s > c$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$	
$e^{ct}f(t)$	$F(s-c)$	
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	$c > 0$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$	
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	
$f(t)$ periodic with period T	$\frac{\int_0^T f(t)e^{-st}\,dt}{1-e^{-sT}}$	