3.3 Optimizing Functions of Several Variables
3.4 Lagrange Multipliers

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Math 20C
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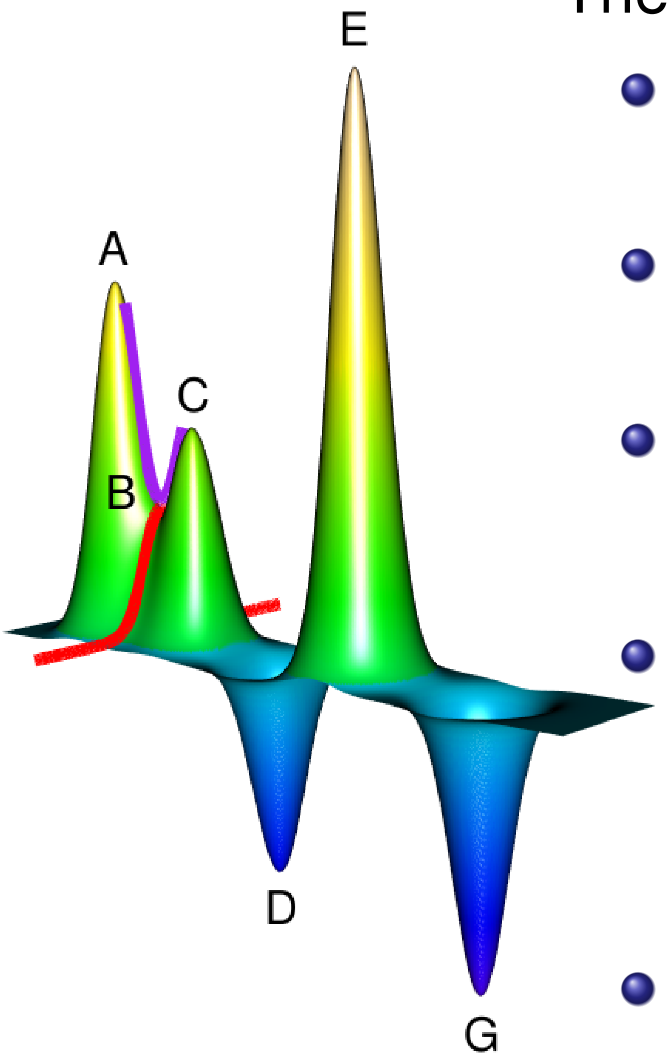
Optimizing $y = f(x)$

- In Math 20A, we found the minimum and maximum of $y = f(x)$ by using derivatives.
- **First derivative:**
  Solve for points where $f'(x) = 0$.
  Each such point is called a *critical point*.
- **Second derivative:**
  For each critical point $x = a$, check the sign of $f''(a)$:
  - $f''(a) > 0$: The value $y = f(a)$ is a local minimum.
  - $f''(a) < 0$: The value $y = f(a)$ is a local maximum.
  - $f''(a) = 0$: The test is inconclusive.
- Also may need to check points where $f(x)$ is defined but the derivatives aren't, as well as boundary points.
- We will generalize this to functions $z = f(x, y)$.

Local extrema (= maxima or minima)

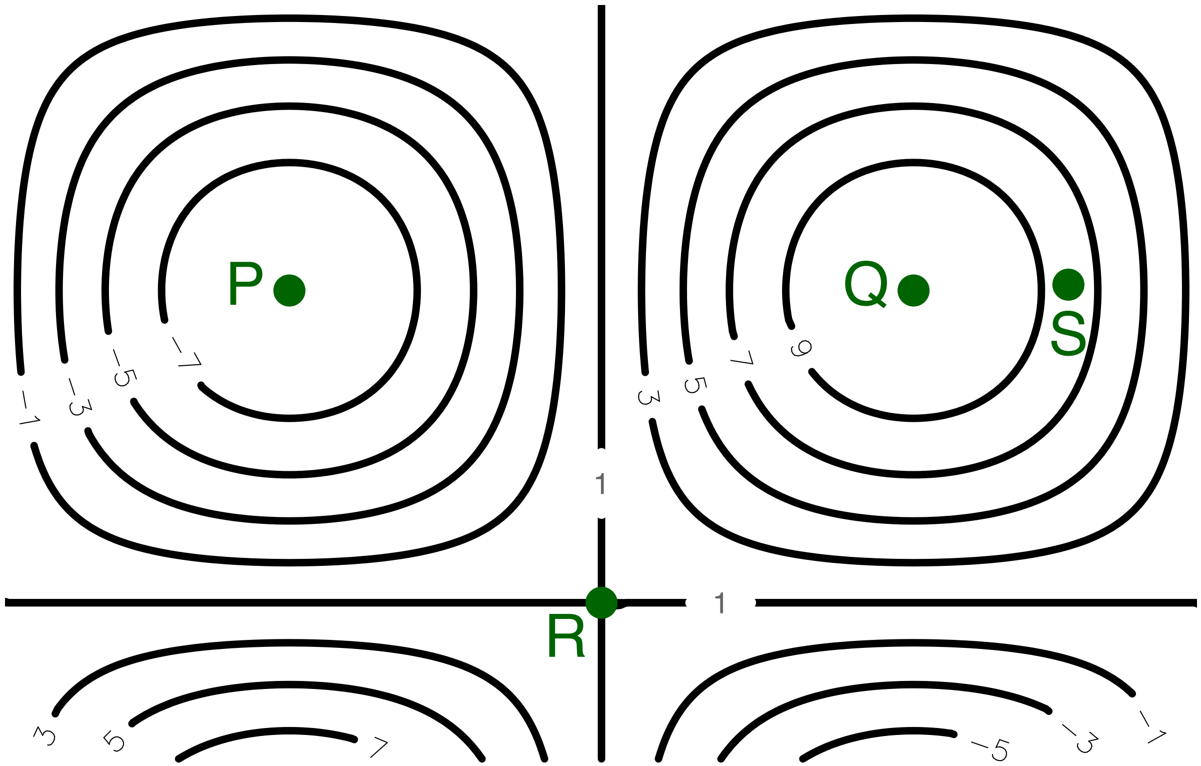


Consider a function $z = f(x, y)$.
The point $(x, y) = (a, b)$ is a

- *local maximum* when $f(x, y) \leqslant f(a, b)$ for all (x, y) in a small disk (filled-in circle) around (a, b);
- *global maximum* (a.k.a. *absolute maximum*) when $f(x, y) \leqslant f(a, b)$ for all (x, y);
- *local minimum* and *global minimum* are similar with $f(x, y) \geqslant f(a, b)$.

- A, C, E are local maxima (plural of maximum)
  E is the global maximum
  D, G are local minima
  G is the global minimum
- B is maximum in the red cross-section but minimum in the purple cross-section!
  It's called a *saddle point*.

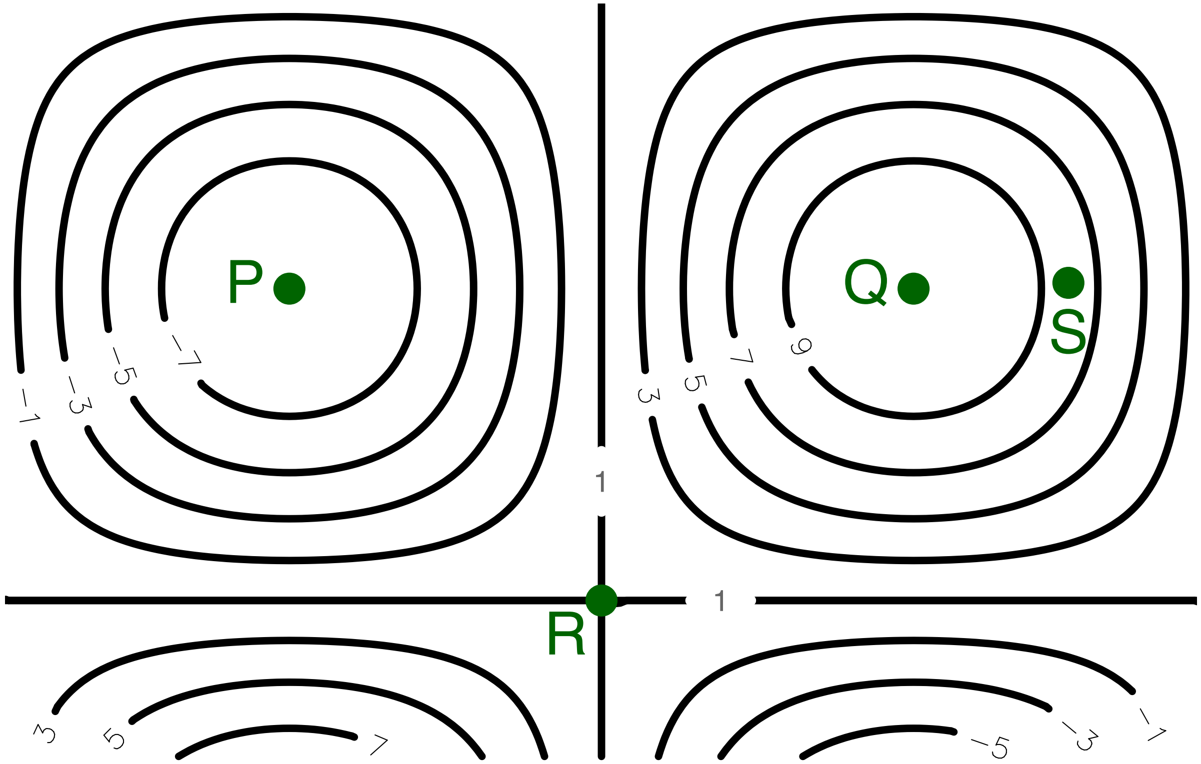
Critical points on a contour map



Classify each point P, Q, R, S as local maximum or minimum, saddle point, or none.

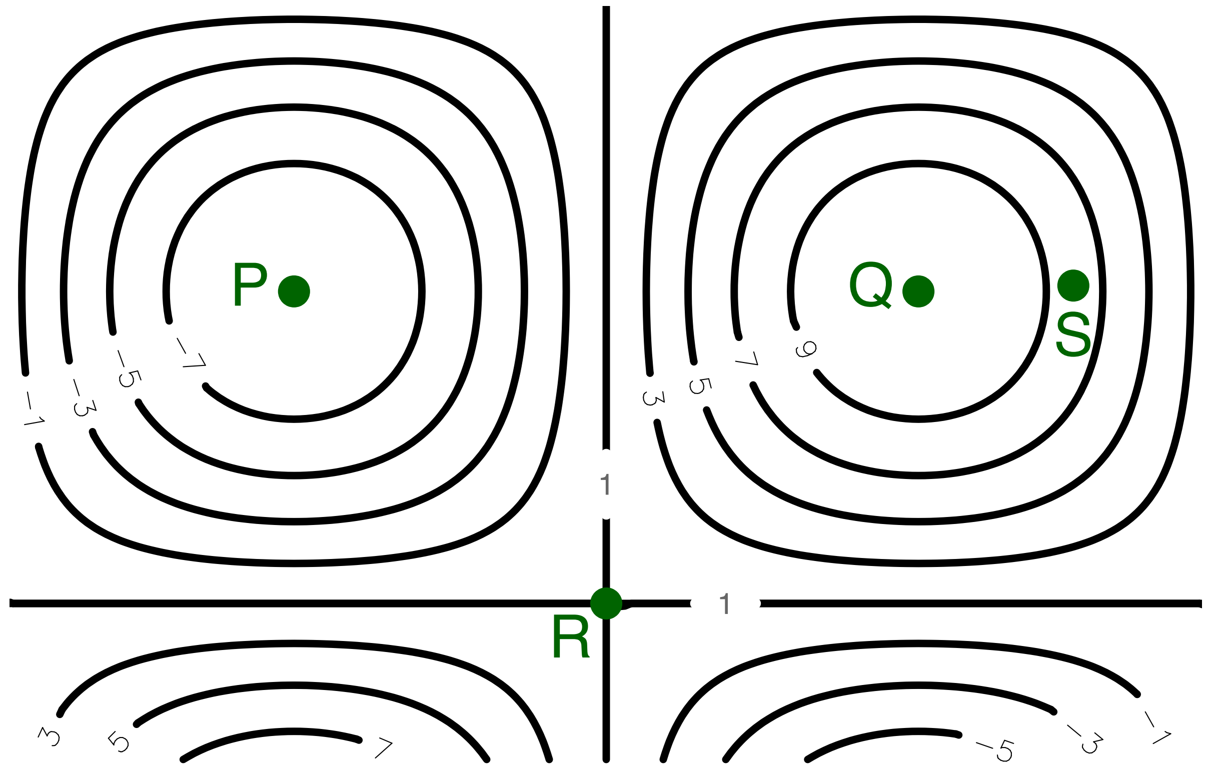
- Isolated max/min usually have small closed curves around them. Values decrease towards P, so P is a local minimum. Values increase towards Q, so Q is a local maximum.

Critical points on a contour map



- The crossing contours have the same value, 1. (If they have different values, the function is undefined at that point.)
- Here, the crossing contours give four regions around R.
- The function has
  - a local min. at R on lines with positive slope (goes from >1 to 1 to >1)
  - a local max. at R on lines with neg. slope (goes from <1 to 1 to <1).
- Thus, R is a saddle point.

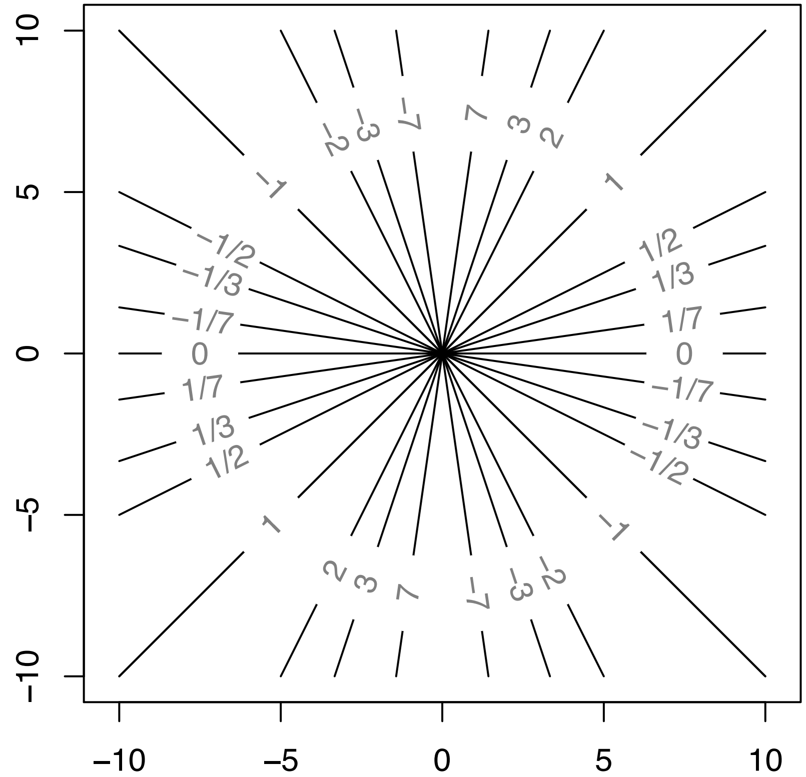
Critical points on a contour map



- S is a regular point.
  Its level curve ≈ 8 is implied but not shown.
  The values are bigger on one side and smaller on the other.

P: local min $\qquad$ Q: local max $\qquad$ R: saddle point $\qquad$ S: none

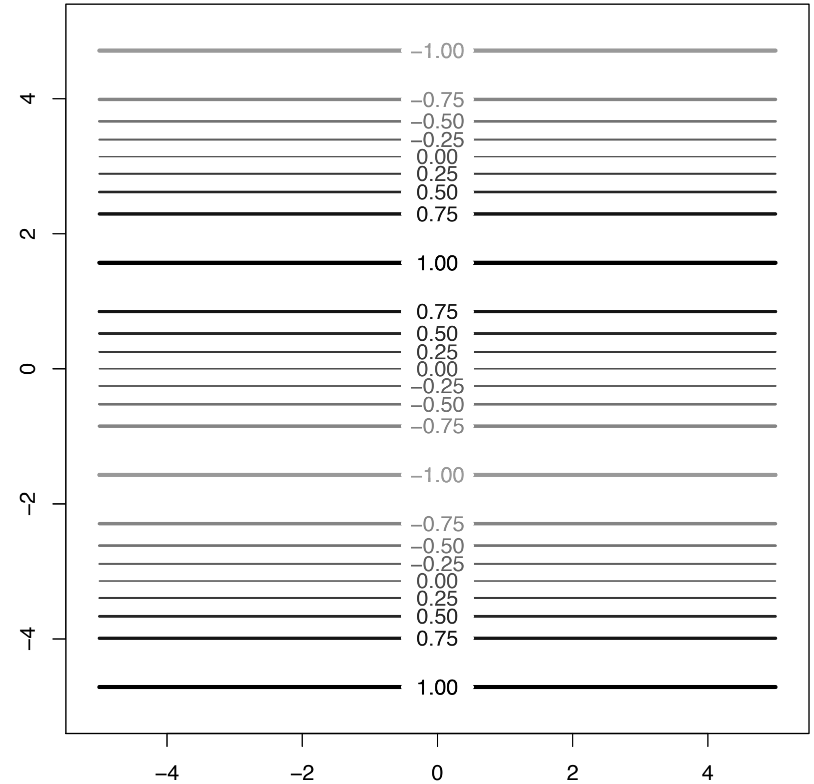
Contour map of $z = y/x$: crossing lines



- Contours of $z = y/x$ are diagonal lines: $z = c$ along $y = cx$.
- Contours cross at $(0, 0)$ and have different values there.
- Function $z = y/x$ is undefined at $(0, 0)$.

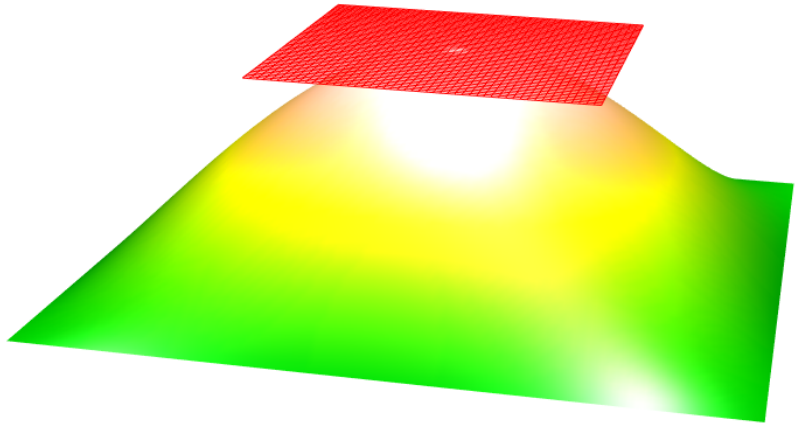
Contour map of $z = \sin(y)$

Minimum and maximum form curves, not just isolated points



- Contours of $z = f(x, y) = \sin(y)$ are horizontal lines $y = \arcsin(z)$
- Maximum at $y = (2k + \frac{1}{2})\pi$ for all integers k
  Minimum at $y = (2k - \frac{1}{2})\pi$
- These are curves, not isolated points enclosed in contours.

Finding the minimum/maximum values of $z = f(x,y)$



- The tangent plane is horizontal at a local minimum or maximum:
  $$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) - z = 0.$$
  The normal vector $\langle\, f_x(a,b), f_y(a,b), -1 \,\rangle \parallel z$-axis
  when $f_x(a,b) = f_y(a,b) = 0$, or $\nabla f(a,b) = \vec{0}$.

- At points where $\nabla f \neq \vec{0}$, we can make $f(x,y)$
  - larger by moving in the direction of ∇f;
  - smaller by moving in the direction of $-\nabla f$.

- (a,b) is a *critical point* if $\nabla f(a,b)$ is $\vec{0}$ or is undefined.
  These are candidates for being maximums or minimums.

- Critical points found in the same way for $f(x,y,z,\ldots)$.

Completing the squares review

- $(x+m)^2 = x^2 + 2mx + m^2$
- For a quadratic $x^2 + bx + c$, take half the coefficient of x:
$$b/2$$
- Form the square:
$$(x + b/2)^2 = x^2 + bx + (b/2)^2$$
- Adjust the constant term:
$$x^2 + bx + c = (x + b/2)^2 + d \qquad \text{where } d = c - (b/2)^2$$

Example: $x^2 + 10x + 13$

- Take half the coefficient of x: $10/2 = 5$
- Expand $(x+5)^2 = x^2 + 10x + 25$
- Add/subtract the necessary constant to make up the difference:
$$x^2 + 10x + 13 = (x+5)^2 - 12$$

Completing the squares review

For $ax^2 + bx + c$, complete the square for $a(x^2 + (b/a)x)$ and then adjust the constant.

Example: $10y^2 - 60y + 8$

- $10y^2 - 60y + 8 = 10(y^2 - 6y) + 8$
- $y^2 - 6y = (y - 3)^2 - 9$
- $10y^2 - 60y + 8 = 10(y - 3)^2 + ?$
- $10(y - 3)^2 = 10(y^2 - 6y + 9) = 10y^2 - 60y + 90$
- $10y^2 - 60y + 8 = 10(y - 3)^2 - 82$

Critical points

- Let $f(x,y) = x^2 - 2x + y^2 - 4y + 15$
  $\nabla f = \langle 2x-2, 2y-4 \rangle$
- $\nabla f = \vec{0}$ at $x = 1$, $y = 2$, so $(1,2)$ is a critical point.
- Use $(x-1)^2 = x^2 - 2x + 1$
  $(y-2)^2 = y^2 - 4y + 4$
  $f(x,y) = (x-1)^2 + (y-2)^2 + 10$

  We "completed the squares": $x^2 - ax = (x - \frac{a}{2})^2 - (\frac{a}{2})^2$
- $f(x,y) \geqslant 10$ everywhere, with global minimum 10 at $(x,y) = (1,2)$.

Second derivative test for functions of two variables

How to classify critical points $\nabla f(a,b) = \vec{0}$ as local minima/maxima or saddle points

Compute all points where $\nabla f(a,b) = \vec{0}$, and classify each as follows:

- Compute the *discriminant* at point (a,b):

$$D = \underbrace{\begin{vmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial y\,\partial x} \\[2ex] \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2} \end{vmatrix}} = f_{xx}(a,b)\,f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

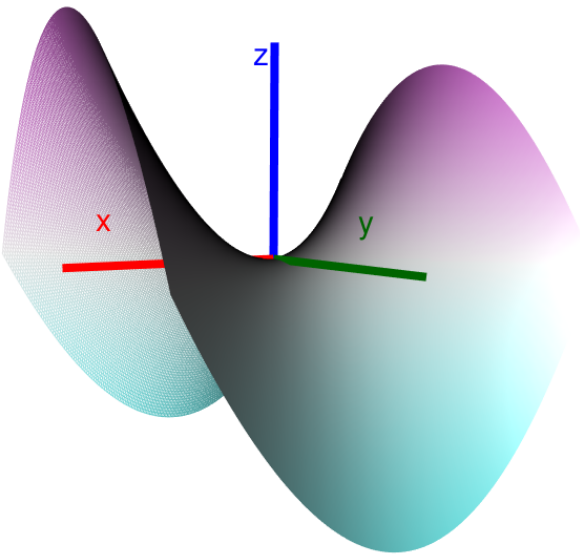
Determinant of “Hessian matrix” at $(x,y) = (a,b)$

- If $D > 0$ and $f_{xx} > 0$ then $z = f(a,b)$ is a local minimum;
  If $D > 0$ and $f_{xx} < 0$ then $z = f(a,b)$ is a local maximum;
  If $D < 0$ then f has a saddle point at (a,b);

  If $D = 0$ then it's inconclusive;
  min, max, saddle, or none of these, are all possible.

Example: $f(x, y) = x^2 - y^2$

Find the critical points of $f(x, y) = x^2 - y^2$ and classify them using the second derivatives test.



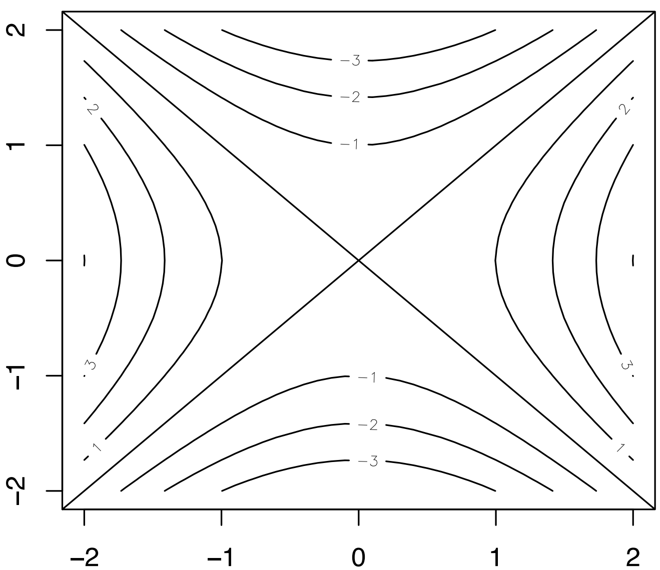
- $\nabla f = \langle 2x, -2y \rangle = \vec{0}$ at $(x, y) = (0, 0)$.
- The $x = 0$ cross-section is $f(0, y) = -y^2 \leqslant 0$.
  The $y = 0$ cross-section is $f(x, 0) = x^2 \geqslant 0$.
  It is neither a minimum nor a maximum.
- $f_{xx}(x, y) = 2$ and $f_{xx}(0, 0) = 2$
  $f_{yy}(x, y) = -2$ and $f_{yy}(0, 0) = -2$
  $f_{xy}(x, y) = 0$ and $f_{xy}(0, 0) = 0$

  $$\begin{aligned} D &= f_{xx}(0,0) f_{yy}(0,0) - (f_{xy}(0,0))^2 \\ &= (2)(-2) - 0^2 = -4 < 0 \end{aligned}$$

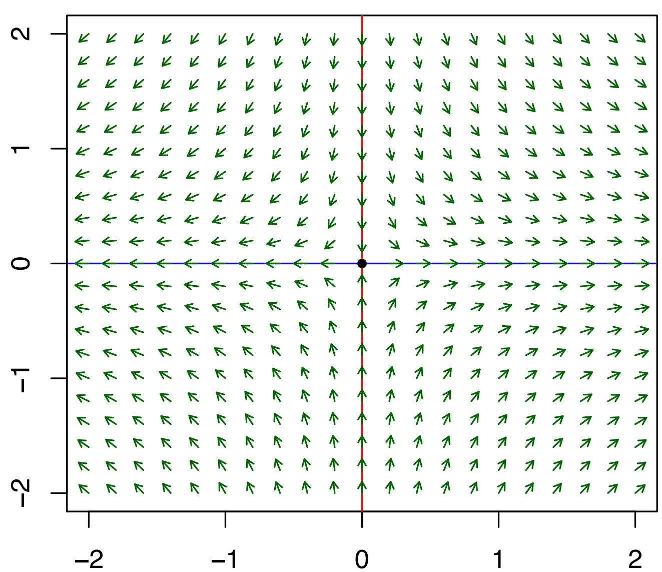
  so $(0, 0)$ is a saddle point

- $\nabla f = \langle 2x, -2y \rangle$ points in the direction of greatest increase of $f(x,y)$.
- The function increases as we move towards the x-axis and away from the y-axis. At the origin, it increases or decreases depending on the direction of approach.

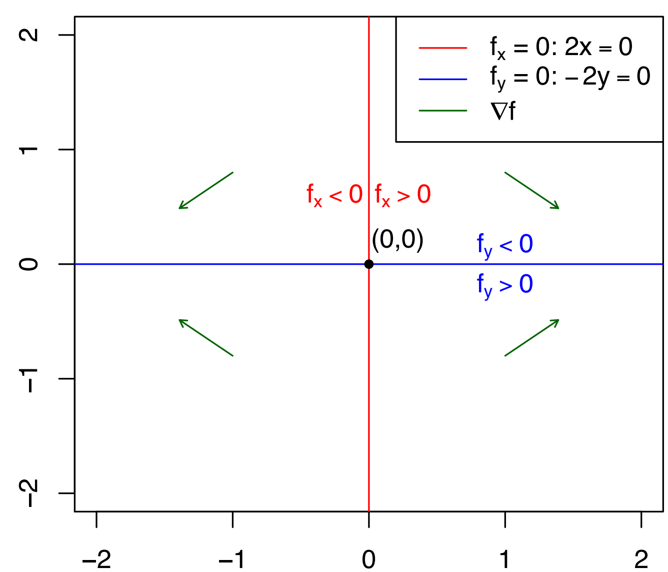
Contour plot



Detailed direction of gradient



General direction of gradient



Example: $f(x,y) = 8y^3 + 12x^2 - 24xy$

Find the critical points of $f(x,y)$ and classify them using the second derivatives test.

- Solve for first derivatives equal to 0:

$$\begin{array}{lll} f_x = 24x - 24y = 0 & \text{gives} & x = y \\ f_y = 24y^2 - 24x = 0 & \text{gives} & 24y^2 - 24y = 24y(y-1) = 0 \\ & \text{so} & y = 0 \text{ or } y = 1 \\ x = y & \text{so} & (x,y) = (0,0) \text{ or } (1,1) \end{array}$$

- Critical points: $(0,0)$ and $(1,1)$
- Second derivative test: $(D = f_{xx}f_{yy} - (f_{xy})^2)$

Crit pt	f	$f_{xx} = 24$	$f_{yy} = 48y$	$f_{xy} = -24$	D	Type
$(0,0)$	0	24	0	-24	-576	$D < 0$ saddle
$(1,1)$	-4	24	48	-24	576	$D > 0$ and $f_{xx} > 0$ local minimum

- No absolute min or max: $f(0,y) = 8y^3$ ranges over $(-\infty, \infty)$

Example: $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$

Find the critical points of $f(x,y)$ and classify them using the second derivatives test.

- Solve for first derivatives equal to 0:

$$f_x = 3x^2 - 3 = 0 \quad \text{gives} \quad x = \pm 1$$
$$f_y = 3y^2 - 6y = 3y(y-2) = 0 \quad \text{gives} \quad y = 0 \text{ or } y = 2$$

- Critical points: $(-1,0)$, $(1,0)$, $(-1,2)$, $(1,2)$
- Second derivative test: $(D = f_{xx} f_{yy} - (f_{xy})^2)$

Crit pt	f	$f_{xx} = 6x$	$f_{yy} = 6y - 6$	$f_{xy} = 0$	D	Type
$(-1,0)$	3	-6	-6	0	36	$D > 0$ and $f_{xx} < 0$: local max
$(1,0)$	-1	6	-6	0	-36	$D < 0$: saddle
$(-1,2)$	-1	-6	6	0	-36	$D < 0$: saddle
$(1,2)$	-5	6	6	0	36	$D > 0$ and $f_{xx} > 0$: local min

Example: $f(x,y) = xy(1-x-y)$

Find the critical points of $f(x,y)$ and classify them.

- Solve for first derivatives equal to 0:

$$f = xy - x^2y - xy^2$$
$$f_x = y - 2xy - y^2 = y(1-2x-y) \quad \text{gives} \quad y=0 \ \text{ or } \ 1-2x-y=0$$
$$f_y = x - x^2 - 2xy = x(1-x-2y) \quad \text{gives} \quad x=0 \ \text{ or } \ 1-x-2y=0$$

- Two solutions of $f_x=0$ and two of $f_y=0$ gives $2\cdot 2=4$ combinations:
  - $y=0$ and $x=0$ gives $(x,y)=(0,0)$.
  - $y=0$ and $1-x-2y=0$ gives $(x,y)=(1,0)$.
  - $1-2x-y=0$ and $x=0$ gives $(x,y)=(0,1)$.
  - $1-2x-y=0$ and $1-x-2y=0$:

    The 1st equation gives $y=1-2x$. Plug that into the 2nd equation:

$$0 = 1-x-2y = 1-x-2(1-2x) = 1-x-2+4x = 3x-1$$

so $x=\frac{1}{3}$ and $y=1-2x=1-2(\frac{1}{3})=\frac{1}{3}$ gives $(x,y)=(\frac{1}{3},\frac{1}{3})$.

Example: $f(x,y) = xy(1-x-y)$

Classify the critical points using the second derivatives test.

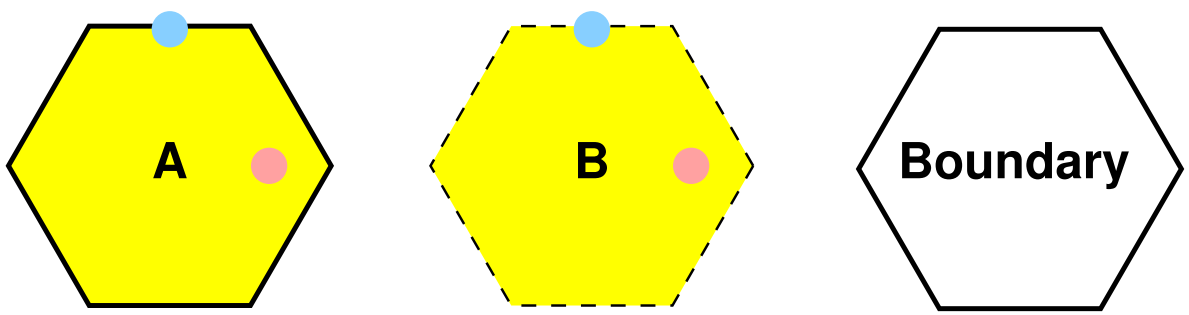
- Derivatives:

$$f = xy - x^2y - xy^2 \qquad f_x = y - 2xy - y^2 \qquad f_y = x - x^2 - 2xy$$
$$f_{xx} = -2y \qquad f_{yy} = -2x$$
$$f_{xy} = f_{yx} = 1 - 2x - 2y$$

- Second derivative test: $(D = f_{xx}f_{yy} - (f_{xy})^2)$

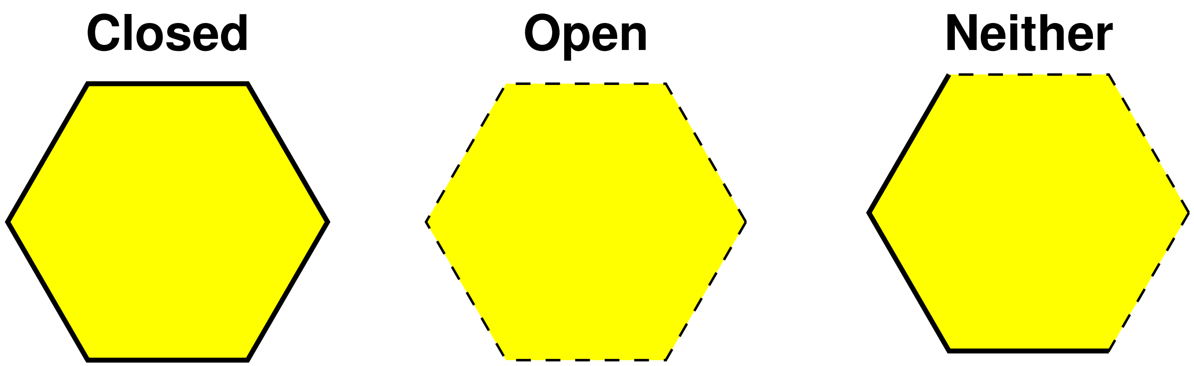
Crit pt	f	f_{xx}	f_{yy}	f_{xy}	D	Type
$(0,0)$	0	0	0	1	-1	$D<0$: saddle
$(1,0)$	0	0	-2	-1	-1	$D<0$: saddle
$(0,1)$	0	-2	0	-1	-1	$D<0$: saddle
$(1/3,1/3)$	$1/27$	$-2/3$	$-2/3$	$-1/3$	$1/3$	$D>0$ and $f_{xx}<0$: local maximum

Boundary of a region



- Consider a region $A \subset \mathbb{R}^n$.
- A point is a *boundary point* of A if every disk (blue) around that point contains some points in A and some points not in A.
- A point is an *interior point* of A if there is a small enough disk (pink) around it fully contained in A.
- In both A and B, the boundary points are the same: the perimeter of the hexagon.
- ∂A denotes the set of boundary points of A.

Extreme Value Theorem



- A region is *bounded* if it fits in a disk of finite radius.
- A region is *closed* if it contains all its boundary points and *open* if every point in it is an interior point.
- Open and closed are *not* opposites: e.g., $\mathbb{R}^2$ is open and closed! The third example above is neither open nor closed.

Extreme Value Theorem

If $f(x, y)$ is continuous on a closed and bounded region, then it has a global maximum and a global minimum within that region.

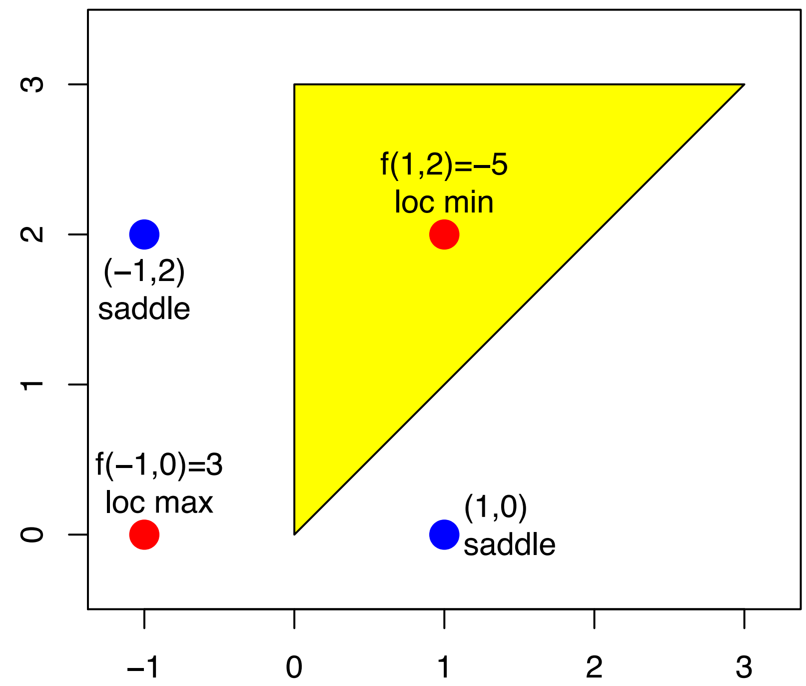
To find these, consider the local minima/maxima of $f(x, y)$ that are within the region, and also analyze the boundary of the region.

Example: $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in a triangle

Find the global minimum and maximum of $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in the triangle with vertices $(0,0)$, $(0,3)$, $(3,3)$.

Critical points inside the region

- First find and classify the critical points of f. (We already did.)
- $f(1,2) = -5$ is a local minimum and is inside the triangle.
- Ignore the other critical points since they're outside the triangle.
- Ignore the saddle points.



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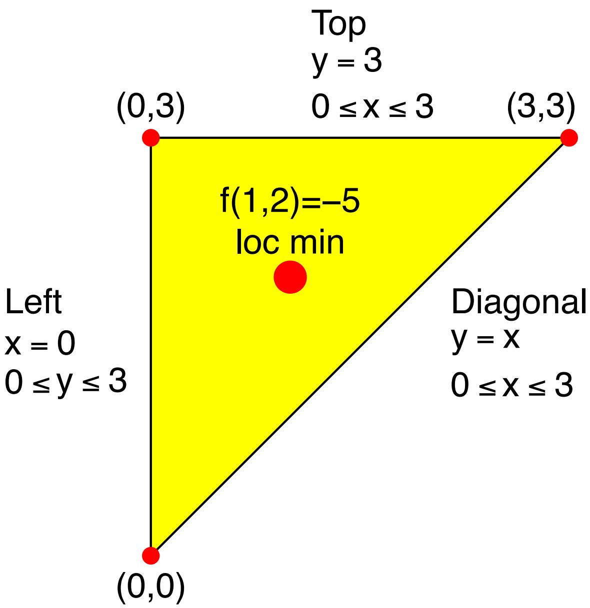
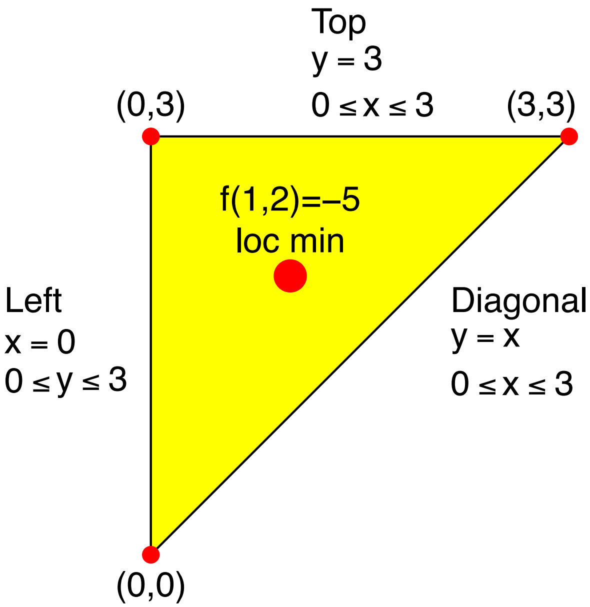


Extrema on left edge: $x = 0$ and $0 \leqslant y \leqslant 3$

- Set
  $$g(y) = f(0,y) = y^3 - 3y^2 + 1 \quad \text{for} \quad 0 \leqslant y \leqslant 3$$
  $$g'(y) = 3y^2 - 6y = 3y(y-2)$$
  $$g'(y) = 0 \quad \text{at } y = 0 \text{ or } 2.$$
- We consider $y = 0$ and 2 by that test.
  We also consider boundaries $y = 0$ and 3.
- Candidates: $f(0,0) = 1$
  $f(0,2) = -3$
  $f(0,3) = 1$
- We could use the second derivatives test for one variable, but we'll do it another way.

Example: $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ in a triangle

Find the global minimum and maximum of $f(x, y) = x^3 + y^3 - 3x - 3y^2 + 1$ in the triangle with vertices $(0, 0)$, $(0, 3)$, $(3, 3)$.

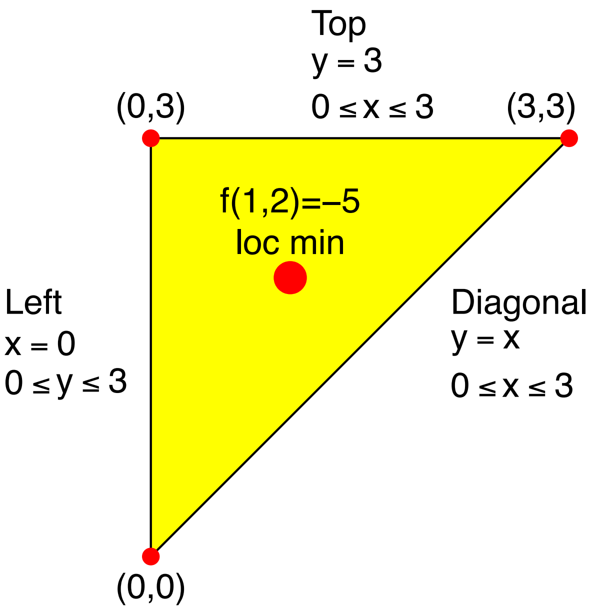


Extrema on top edge: $y = 3$ and $0 \leqslant x \leqslant 3$

- Set
$$h(x) = f(x, 3) = x^3 + 27 - 3x - 27 + 1$$
$$= x^3 - 3x + 1 \quad \text{for} \quad 0 \leqslant x \leqslant 3$$
$$h'(x) = 3x^2 - 3$$
$h'(x) = 0$ at $x = \pm 1$ (but -1 is out of range)
- Also consider the boundaries $x = 0$ and 3.
- Candidates:
$$f(0, 3) = 1$$
$$f(1, 3) = -1$$
$$f(3, 3) = 19$$

Example: $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in a triangle

Find the global minimum and maximum of $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in the triangle with vertices $(0,0)$, $(0,3)$, $(3,3)$.



Diagonal edge: $y = x$ for $0 \leqslant x \leqslant 3$

- For $0 \leqslant x \leqslant 3$, set

$$\begin{aligned} p(x) = f(x,x) &= 2x^3 - 3x - 3x^2 + 1 \\ &= 2x^3 - 3x^2 - 3x + 1 \end{aligned}$$

$$p'(x) = 6x^2 - 6x - 3$$

$$p'(x) = 0 \quad \text{at } x = \tfrac{1 \pm \sqrt{3}}{2} \approx -0.366, 1.366$$

(but $\frac{1-\sqrt{3}}{2}$ is out of range)

- Also consider the boundaries $x = 0$ and 3.
- Candidates: $f(0,0) = 1$

$$f(3,3) = 19$$

$$f\left(\tfrac{1+\sqrt{3}}{2}, \tfrac{1+\sqrt{3}}{2}\right) = -1 - \tfrac{3\sqrt{3}}{2} \approx -3.598$$

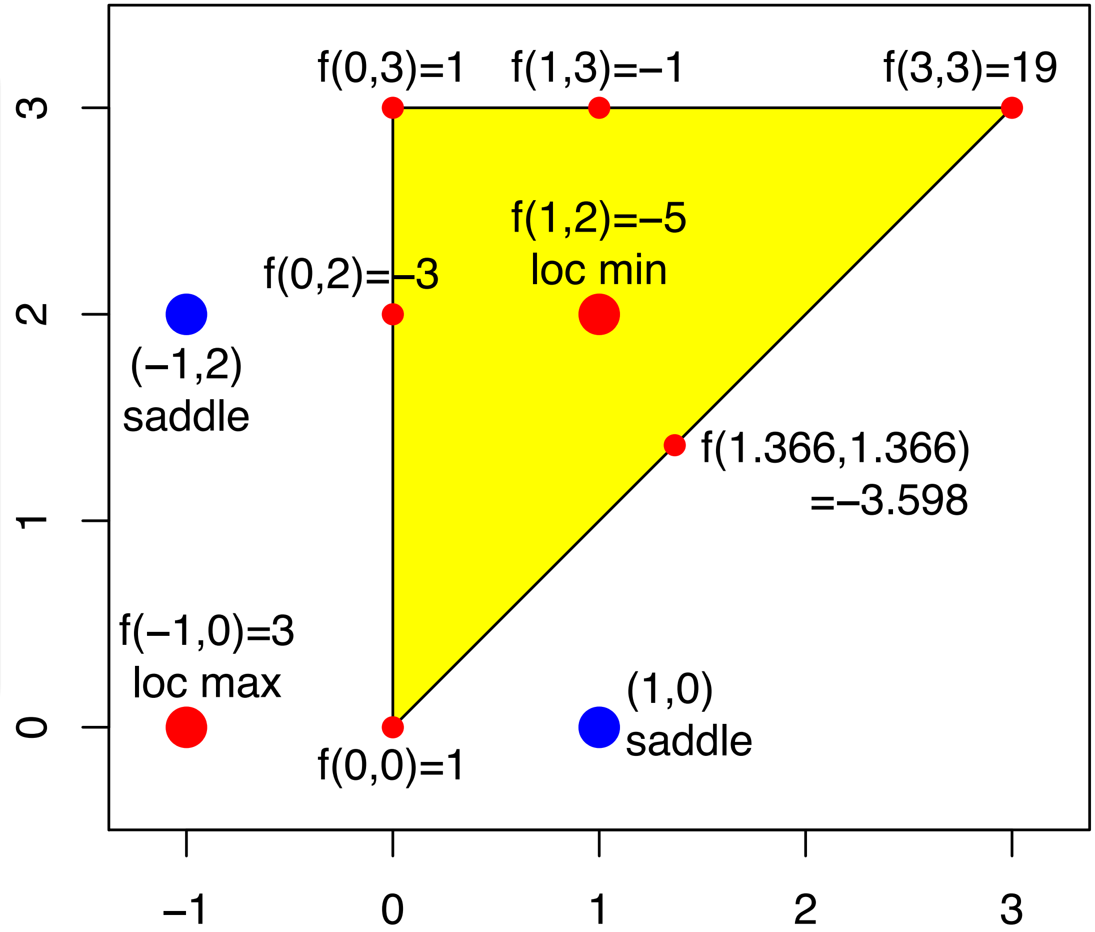
Example: $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in a triangle

Find the global minimum and maximum of $f(x,y) = x^3 + y^3 - 3x - 3y^2 + 1$ in the triangle with vertices $(0,0)$, $(0,3)$, $(3,3)$.

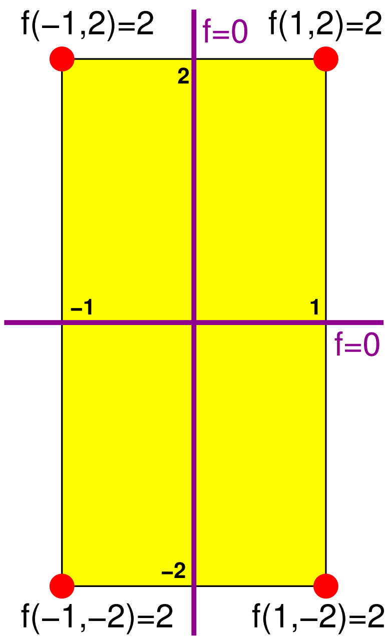
Compare all candiate points

$\boldsymbol{f(1,2) = -5}$**:**
The global minimum is -5.
It occurs at $(x,y) = (1,2)$.

$\boldsymbol{f(3,3) = 19}$**:**
The global maximum is 19.
It occurs at $(x,y) = (3,3)$.



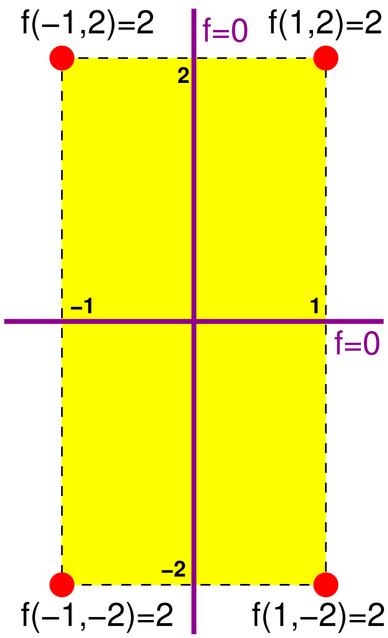
Extrema of $f(x, y) = |xy|$: ∇f isn't defined everywhere



Extrema of $f(x, y) = |xy|$ on rectangle $-1 \leqslant x \leqslant 1$, $-2 \leqslant y \leqslant 2$

- 1st & 3rd quadrants: $f(x, y) = xy$ and $\nabla f = \langle y, x \rangle$.
- 2nd & 4th quadrants: $f(x, y) = -xy$ and $\nabla f = -\langle y, x \rangle$.
- Away from the axes, $\nabla f \neq \vec{0}$.
- On the axes, ∇f is undefined.
  - $f(x, 0) = f(0, y) = 0$ on the axes.
    All points on the axes are tied for global minimum.
- On the perimeter, $f(\pm 1, y) = |y|$ and $f(x, \pm 2) = 2|x|$:
  - Minimum $f = 0$ at $(\pm 1, 0)$ and $(0, \pm 2)$.
  - Maximum $f = 2$ at $(1, 2), (1, -2), (-1, 2), (-1, -2)$.
- The global maximum is $f = 2$ at $(1, 2), (1, -2), (-1, 2), (-1, -2)$.

Extrema of $f(x,y)=|xy|$: ∇f isn't defined everywhere



Extrema of $f(x,y)=|xy|$ on open rectangle $-1<x<1,\ -2<y<2$

- Global minimum is still $f=0$ on axes.
- No global maximum. While $f(x,y)$ gets arbitrarily close to 2, it never reaches 2 since those corners are not in the open rectangle.

Optional: Second derivative test for $f(x, y, z, \ldots)$

Full coverage requires Linear Algebra (Math 18)

- The *Hessian matrix* of $f(x, y, z)$ is

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\,\partial x} & \frac{\partial^2 f}{\partial z\,\partial x} \\ \frac{\partial^2 f}{\partial x\,\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\,\partial y} \\ \frac{\partial^2 f}{\partial x\,\partial z} & \frac{\partial^2 f}{\partial y\,\partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

- For $f : \mathbb{R}^n \to \mathbb{R}$, it's an $n \times n$ matrix of 2nd partial derivatives.
- For each point with $\nabla f = \vec{0}$, compute the determinants of the upper left 1×1, 2×2, 3×3, $\ldots$, $n \times n$ submatrices.
  - If the $n \times n$ determinant is zero, the test is inconclusive.
  - If the determinants are all positive, it's a local minimum.
  - If signs of determinants alternate $-, +, -, \ldots$, it's a local maximum.
  - Otherwise, it's a saddle point.

  - We did 2×2 and 3×3 determinants. For 1×1, $\det[x] = x$.
    $n \times n$ determinants are covered in Linear Algebra (Math 18).

Optional example: $f(x,y,z) = x^2 + y^2 + z^2 + 2xyz + 10$

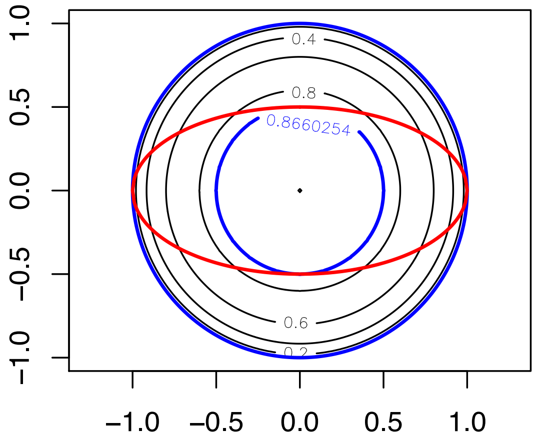
- Solve $\nabla f = \vec{0}$: $$\nabla f = \langle 2x + 2yz, 2y + 2xz, 2z + 2xy \rangle = \vec{0}$$ $$x = -yz, \quad y = -xz, \quad z = -xy.$$
- There are five solutions (x,y,z) of $\nabla f = \vec{0}$ (work not shown): $$(0,0,0),\ (1,1,-1),\ (-1,1,1),\ (1,-1,1),\ (-1,-1,-1).$$
- Hessian $= \begin{bmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{bmatrix}$ At $(0,0,0)$: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $$\det\begin{bmatrix}2\end{bmatrix} = 2 \quad \det\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4 \quad \det\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 8$$
- All positive, so $f(0,0,0) = 10$ is a local minimum.

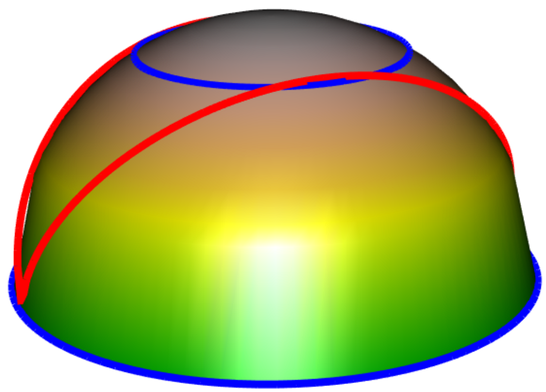
Optional example: $f(x,y,z) = x^2 + y^2 + z^2 + 2xyz + 10$

- Hessian $= \begin{bmatrix} 2 & 2z & 2y \\ 2z & 2 & 2x \\ 2y & 2x & 2 \end{bmatrix}$ At $(1,1,-1)$: $\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

  det $\begin{bmatrix} 2 \end{bmatrix} = 2$ det $\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 0$ det $\begin{bmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = -32$
- Signs $+$, 0, $-$, so saddle point.
- Critical points $(-1,1,1)$, $(1,-1,1)$, $(-1,-1,-1)$ give the same determinants $2, 0, -32$ as this case, so they're also saddle points.

Optimization with a constraint





- A hiker hikes on a mountain $z = f(x, y) = \sqrt{1 - x^2 - y^2}$.
- Plot their trail on a topographic map: $x^2 + 4y^2 = 1$ (red ellipse).
- What is the minimum and maximum height reached, and where?
- On the ellipse, $y^2 = (1 - x^2)/4$ and $-1 \leqslant x \leqslant 1$, so

$$z = \sqrt{1 - x^2 - (1 - x^2)/4} = \sqrt{\tfrac{3}{4}(1 - x^2)}$$

Minimum at $x = \pm 1$

- $y^2 = (1 - (\pm 1)^2)/4 = 0$ so $y = 0$
- $z = \sqrt{(3/4)(1 - (\pm 1)^2)} = 0$
- Min: $z = 0$ at $(x, y) = (\pm 1, 0)$

Maximum at $x = 0$

- $y^2 = (1 - 0^2)/4 = 1/4$ so $y = \pm\frac{1}{2}$
- $z = \sqrt{\frac{3}{4}(1 - 0^2)} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$
- Max: $z = \frac{\sqrt{3}}{2}$ at $(x, y) = (0, \pm\frac{1}{2})$

3.4. Lagrange Multipliers

General problem

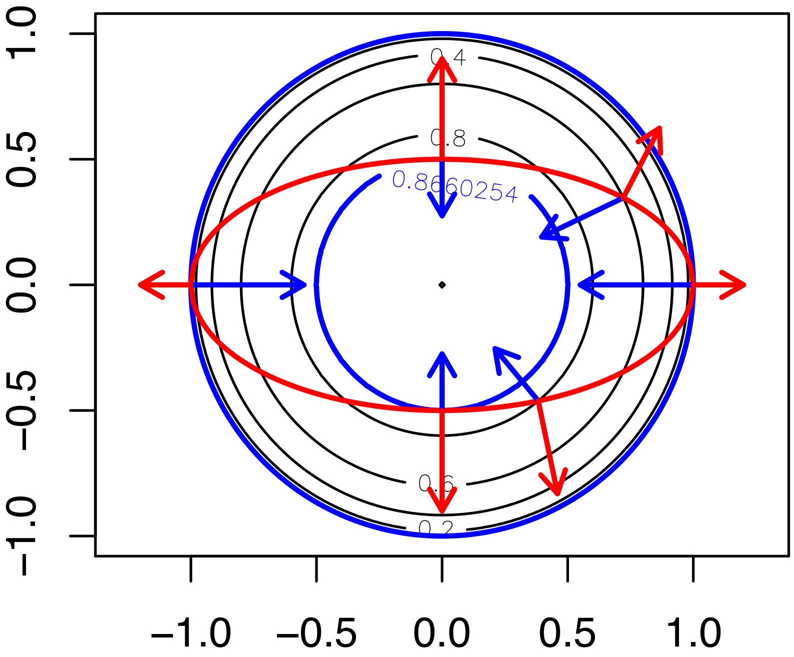
Find the minimum and maximum of $f(x, y, z, \ldots)$
subject to the constraint $g(x, y, z, \ldots) = c$ (constant)

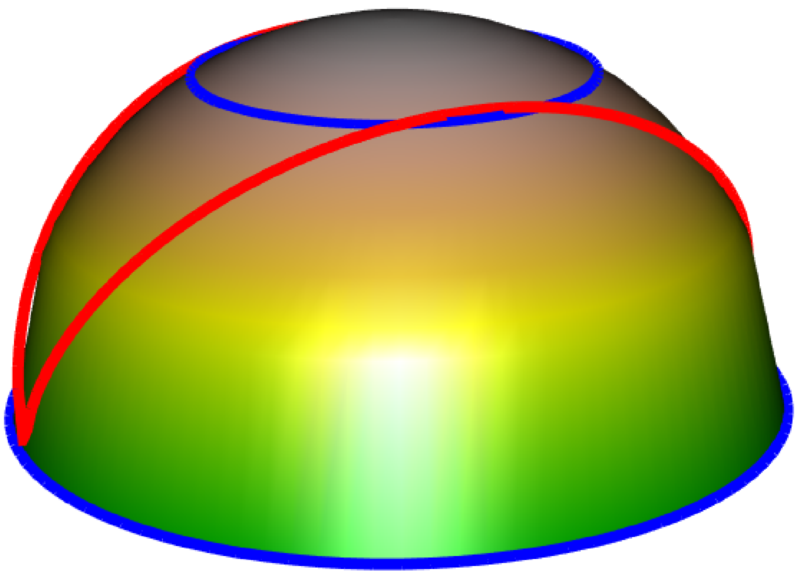
This problem

Find the minimum and maximum of $f(x, y) = \sqrt{1 - x^2 - y^2}$
subject to the constraint $g(x, y) = x^2 + 4y^2 = 1$

Approaches

- Use the constraint g to solve for one variable in terms of the other(s), then plug into f and find its extrema.
- New method: Lagrange Multipliers





- On the contour map, when the trail ($g(x,y) = c$, in red) crosses a contour of $f(x,y)$, f is lower on one side and higher on the other.
- The min/max of $f(x,y)$ on the trail occurs when the trail is tangent to a contour of $f(x,y)$! The trail goes up to a max and then back down, staying on the same side of the contour of f.
- Recall $\nabla f \perp$ contours of f $\qquad$ $\nabla g \perp$ contours of g
  So contours of f and g are tangent when $\nabla f \| \nabla g$, or $\nabla f = \lambda \nabla g$ for some scalar λ (called a *Lagrange Multiplier*).

Lagrange Multipliers for the ellipse path

- Find the minimum and maximum of $z = \sqrt{1 - x^2 - y^2}$
  subject to the constraint $x^2 + 4y^2 = 1$.
- This is equivalent to finding the extrema of $z^2 = 1 - x^2 - y^2$.
- Set $f(x, y) = 1 - x^2 - y^2$ and $g(x, y) = x^2 + 4y^2$ (constraint: $= 1$).
  $\nabla f = \langle -2x, -2y \rangle$ $\qquad$ $\nabla g = \langle 2x, 8y \rangle$
- Solve $\nabla f = \lambda \nabla g$ and $g(x, y) = c$ for x, y, λ:

$$-2x = 2\lambda x \qquad -2y = 8\lambda y \qquad x^2 + 4y^2 = 1$$

$$2x(1 + \lambda) = 0 \qquad y(2 + 8\lambda) = 0$$

$x = 0$ or $\lambda = -1$ $\qquad$ $y = 0$ or $\lambda = -1/4$

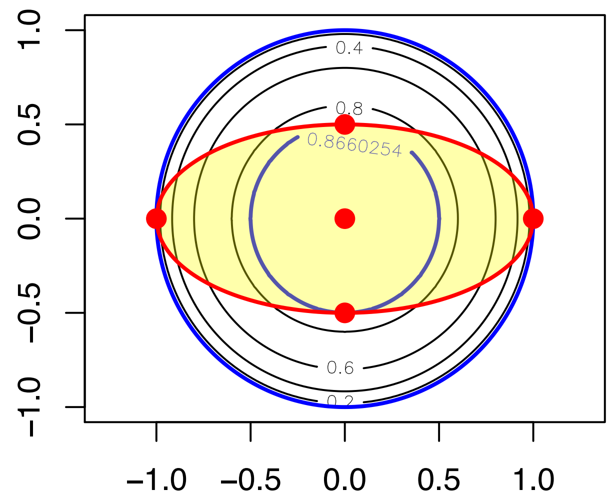
- Solutions:
  - $x = 0$ gives $y = \pm\sqrt{1 - 0^2}/2 = \pm\frac{1}{2}$, $\quad \lambda = -2/8 = -1/4$,
    $z = \sqrt{1 - 0^2 - (1/2)^2} = \sqrt{3}/2$.
  - $\lambda = -1$ gives $y = 0$, $\quad x = \pm\sqrt{1 - 4(0)^2} = \pm 1$,
    $z = \sqrt{1 - (\pm 1)^2 - 0^2} = 0$.

Lagrange Multipliers for the ellipse path

- $\sqrt{1-x^2-y^2}$ is continuous along the closed path $x^2+4y^2=1$, so
  - $z=\frac{\sqrt{3}}{2}$ at $(x,y)=(0,\pm\frac{1}{2})$ are absolute maxima
  - $z=0$ at $(x,y)=(\pm 1,0)$ are absolute minima
- λ is a tool to solve for the extremal points; its value isn't important.

Lagrange Multipliers on Closed Region with Boundary

Find the extrema of $z = \sqrt{1 - x^2 - y^2}$
subject to the constraint $x^2 + 4y^2 \leqslant 1$.



- Analyze interior points and boundary points separately.
Then select the minimum and maximum out of all candidates.

- In $x^2 + 4y^2 < 1$ (yellow interior), use critical points to show the maximum is $f(0, 0) = 1$.

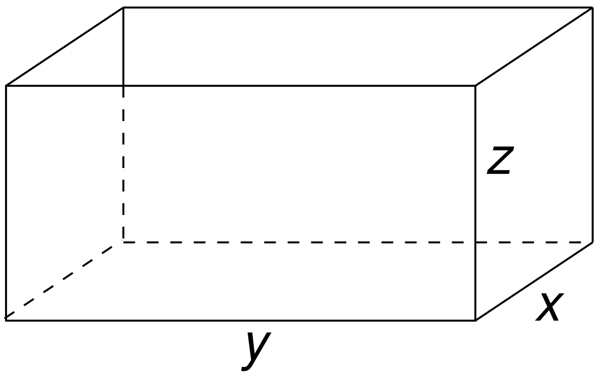
- On boundary $x^2 + 4y^2 = 1$ (red ellipse), use Lagrange Multipliers.
minimum $f(\pm 1, 0) = 0$, maximum $f(0, \pm\frac{1}{2}) = \frac{\sqrt{3}}{2} \approx 0.866$.

- Comparing candidates (red spots) gives
absolute minimum $f(\pm 1, 0) = 0$, absolute maximum $f(0, 0) = 1$.

Example: Rectangular box

Method 1: Critical points

An open rectangular box (5 sides but no top) has volume 500 cm^3.
What dimensions give the minimum surface area, and what is that area?



Volume $V = xyz = 500$

Area bottom + left & right + front & back

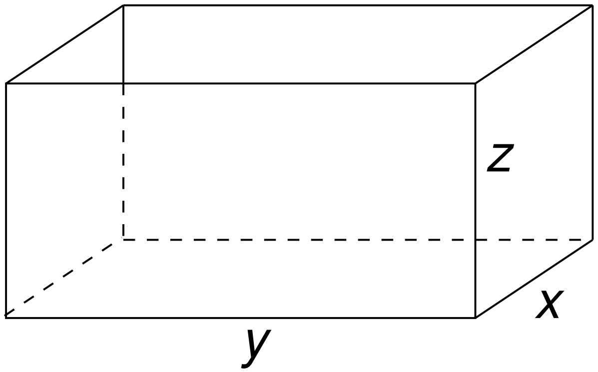
$A = xy + 2xz + 2yz$

- Physical intuition says there is some minimum amount of material needed in order to hold a given volume. We will solve for this.
- There's no maximum, though:
  e.g., let $x = y$, $z = \frac{500}{xy} = \frac{500}{x^2}$, and let $x \to \infty$. Then $A \to \infty$.

Example: Rectangular box

Method 1: Critical points

An open rectangular box (5 sides but no top) has volume 500 cm^3.
What dimensions give the minimum surface area, and what is that area?



Dimensions	$x, y, z > 0$
Volume	$V = xyz = 500$
Area	$A = xy + 2xz + 2yz$

- The volume equation gives $z = \frac{500}{xy}$
- Plug that into the area equation:

$$A = xy + 2x \cdot \frac{500}{xy} + 2y \cdot \frac{500}{xy} = xy + \frac{1000}{y} + \frac{1000}{x}$$

Example: Rectangular box

Method 1: Critical points

$$A = xy + \frac{1000}{y} + \frac{1000}{x}$$

- Find first derivatives:

$$A_x = y - \frac{1000}{x^2} \qquad\qquad A_y = x - \frac{1000}{y^2}$$

- Solve $A_x = A_y = 0$: Plug $y = 1000/x^2$ into $x = 1000/y^2$ to get

$$x = \frac{1000}{(1000/x^2)^2} = \frac{x^4}{1000} \qquad x^4 - 1000x = 0 \qquad x(x^3 - 1000) = 0$$

so $x = 0$ or $x = 10$ (and two complex solutions)

- $x = 0$ violates $V = xyz = 500$.
Also, we need $x > 0$ for a real box.

- $x = 10$ gives $y = \frac{1000}{x^2} = \frac{1000}{10^2} = 10$ and $z = \frac{500}{xy} = \frac{500}{(10)(10)} = 5$

Example: Rectangular box

Method 1: Critical points

$$A = xy + \frac{1000}{y} + \frac{1000}{x}$$

- Check if $x = y = 10$ is a critical point:

$$A_x = y - \frac{1000}{x^2} = 10 - \frac{1000}{10^2} = 10 - 10 = 0$$

$$A_y = x - \frac{1000}{y^2} = 10 - \frac{1000}{10^2} = 10 - 10 = 0$$

- Yes, it's a critical point.
- Solution of original problem:

Dimensions	$x = y = 10$ cm, $z = 5$ cm
Volume	$V = xyz = (10)(10)(5) = 500$ cm^3
Area	$A = xy + 2xz + 2yz$ $= (10)(10) + 2(10)(5) + 2(10)(5) = 300$ cm^2

Example: Rectangular box

Method 1: Critical points

$$A = xy + \frac{1000}{y} + \frac{1000}{x}$$

Second derivatives test at $(x, y) = (10, 10)$:

$$A_{xx} = \frac{2000}{x^3} = \frac{2000}{10^3} = 2$$

$$A_{yy} = \frac{2000}{y^3} = \frac{2000}{10^3} = 2$$

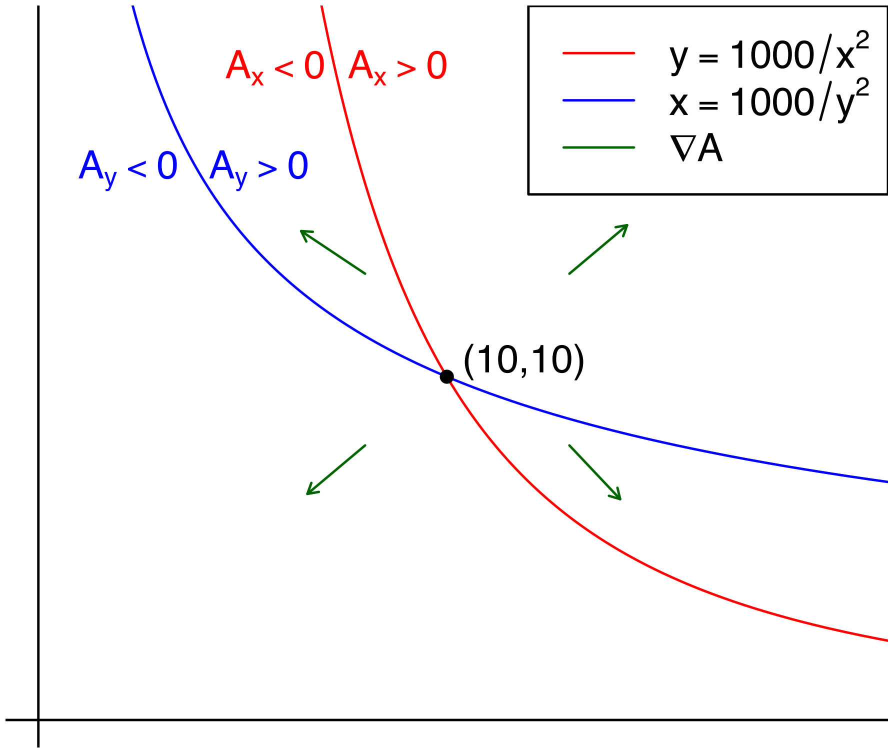
$$A_{xy} = 1$$

$D = (2)(2) - 1^2 = 3 > 0$ and $A_{xx} > 0$ so local minimum

Example: Rectangular box

Method 1: Critical points   Using gradients instead of 2nd derivatives test

$$A = xy + \frac{1000}{y} + \frac{1000}{x} \qquad A_x = y - \frac{1000}{x^2} \qquad A_y = x - \frac{1000}{y^2}$$

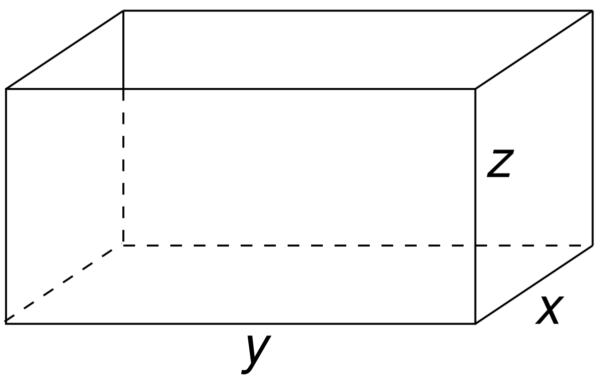


- The signs of A_x, A_y split the first quadrant into four regions.
- $\nabla A(x, y)$ points away from $(10, 10)$ in each region.
- $A(x, y)$ increases as we move away from $(10, 10)$ in each region.
- So $(10, 10)$ is the location of the global minimum.

Example: Rectangular box

Method 2: Lagrange Multipliers

An open rectangular box (5 sides but no top) has volume 500 cm^3. What dimensions give the minimum surface area, and what is that area?



Dimensions	$x, y, z > 0$
Volume	$V = xyz = 500$
Area	$A = xy + 2xz + 2yz$

- Solve $\nabla A = \lambda \, \nabla V$ and $V = xyz = 500$ for x, y, z, λ.
- Solve $\langle y + 2z, x + 2z, 2x + 2y \rangle = \lambda \, \langle yz, xz, xy \rangle$ and $V = xyz = 500$
- Solve for λ:

$$\lambda = \frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy}$$

$$\lambda = \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

There is no division by 0 since $xyz = 500$ implies $x, y, z \neq 0$.

Example: Rectangular box

Method 2: Lagrange Multipliers

$$\lambda = \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x}$$

- Taking any two of those at a time gives

$$\frac{1}{z} = \frac{2}{y} = \frac{2}{x} \qquad \text{so } x = y = 2z.$$

- Combine with $xyz = 500$: $(2z)(2z)(z) = 4z^3 = 500$
  $z^3 = 500/4 = 125$ and $z = 5$
  $x = y = 2z = 10$
  $(x, y, z) = (10, 10, 5)$ cm
- Area: $(10)(10) + 2(10)(5) + 2(10)(5) = 300$ cm^2.
- This method doesn't tell you if it's a minimum or a maximum! Use your intuition (in this case, there is a minimum area that can encompass the volume, but not a maximum) or test nearby values.

Example: Rectangular box

Method 2: Lagrange Multipliers

This method doesn't tell you if it's a minimum or a maximum!

- Use your intuition (in this case, there is a minimum area that can encompass the volume, but not a maximum) or test nearby values.
- Surface $xyz = 500$ (with $x, y, z > 0$) is not bounded, so Extreme Value Theorem doesn't apply. No guarantee there's a global min/max in the region.
- Only one candidate point, so we can't compare candidates.
- Pages 197–201 extend the 2$^{\text{nd}}$ derivatives test to constraint equations, but it uses Linear Algebra (Math 18).

Example: Function of 10 variables

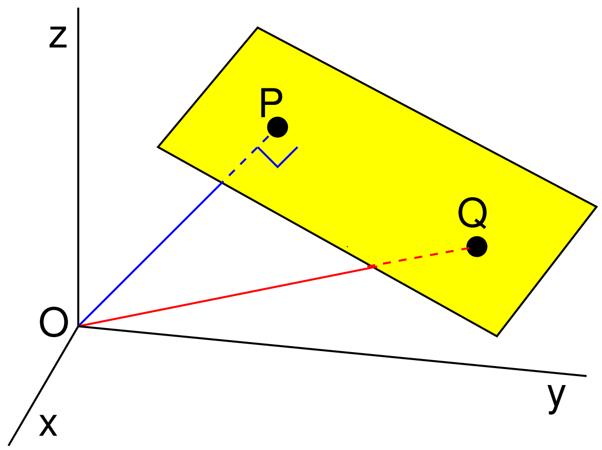
Find 10 positive #'s whose sum is 1000 and whose product is maximized:

Maximize $f(x_1,\ldots,x_{10}) = x_1\,x_2\,\ldots\,x_{10}$ $\qquad \nabla f = \left\langle \frac{f}{x_1},\ldots,\frac{f}{x_{10}} \right\rangle$

Subject to $g(x_1,\ldots,x_{10}) = x_1 + \cdots + x_{10} = 1000$ $\qquad \nabla g = \langle 1,\ldots,1\rangle$

- Solve $\nabla f = \lambda\,\nabla g$: $\frac{f}{x_1} = \cdots = \frac{f}{x_{10}} = \lambda \cdot 1$
  $$x_1 = \cdots = x_{10}$$
- Combine with constraint $g = x_1 + \cdots + x_{10} = 1000$:
  $$10\,x_1 = 1000 \quad \text{so} \quad x_1 = \cdots = x_{10} = 100$$
- The product is $100^{10} = 10^{20}$. This turns out to be the maximum.
- Minimum: as any of the variables approach 0, the product approaches 0, without reaching it. So, in the domain $x_1,\ldots,x_{10} > 0$, the minimum does not exist.

Closest point on a plane to the origin

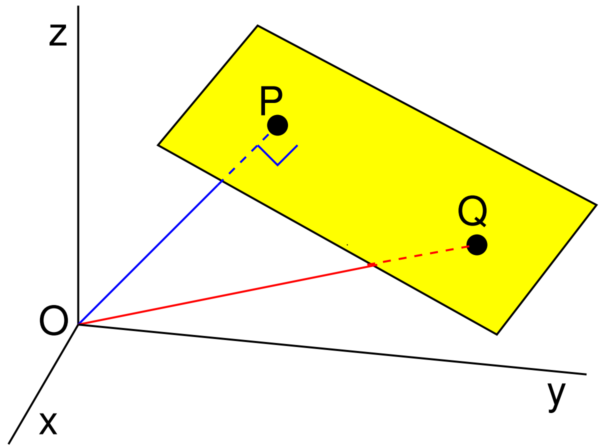


What point on the plane $x + 2y + z = 4$ is closest to the origin?

- Physical intuition tells us there is a minimum but not a maximum.
- No max: plane has infinite extent, with points arbitrarily far away.
- Approaches: vector projections (Chapter 1.2), critical points (3.3), and Lagrange Multipliers (3.4).
- **Generalization**: Given a point A, find the closest point to A on surface $z = f(x, y)$.

Closest point on a plane to the origin

Method 1: Projection



What point on the plane $x + 2y + z = 4$ is closest to the origin?

- Pick *any* point Q on the plane; let's use $Q = (1, 1, 1)$.
- Form the projection of $\vec{a} = \overrightarrow{OQ} = \langle 1, 1, 1 \rangle$ along the normal vector $\vec{n} = \langle 1, 2, 1 \rangle$ to get $\overrightarrow{OP}$, where P is the closest point:

$$\overrightarrow{OP} = \frac{(\vec{a} \cdot \vec{n})\vec{n}}{\|\vec{n}\|^2} = \frac{(1 \cdot 1 + 1 \cdot 2 + 1 \cdot 1)\vec{n}}{1^2 + 2^2 + 1^2} = \frac{4\vec{n}}{6} = \left\langle \frac{2}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle$$

- Closest point is $P = O + \overrightarrow{OP} = (\frac{2}{3}, \frac{4}{3}, \frac{2}{3})$.

Closest point on a plane to the origin

Method 2: Critical points

What point on the plane $x + 2y + z = 4$ is closest to the origin?

- For (x, y, z) on the plane, the distance to the origin is

$$f(x, y, z) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

- This is minimized at the same place as its square:

$$g(x, y, z) = x^2 + y^2 + z^2$$

- On the plane, $z = 4 - x - 2y$. So find (x, y) that minimize

$$h(x, y) = x^2 + y^2 + (4 - x - 2y)^2$$

Then plug the solution(s) of (x, y) into $z = 4 - x - 2y$.

Closest point on a plane to the origin

Method 2: Critical points

What point on the plane $x+2y+z=4$ is closest to the origin?

- Minimize $h(x,y)=x^2+y^2+(4-x-2y)^2$.
- First derivatives:

$$h_x = 2x-2(4-x-2y) = 4x+4y-8$$
$$h_y = 2y+2(-2)(4-x-2y) = 4x+10y-16$$

- Critical points: solve $h_x=h_y=0$:

$$\begin{array}{lll} h_x=0 & \text{gives} & y=2-x \\ h_y=0 & \text{becomes} & 4x+10(2-x)-16 \\ & & \quad =4x+20-10x-16=-6x+4=0 \\ & \text{so} & x=2/3 \text{ and } y=2-2/3=4/3 \end{array}$$

- This gives $z=4-x-2y=4-(2/3)-2(4/3)=2/3$.
- The point is $\boxed{\left(\frac{2}{3},\frac{4}{3},\frac{2}{3}\right)}$.

  Its distance to the origin is $\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{4}{3}\right)^2+\left(\frac{2}{3}\right)^2}=\frac{\sqrt{24}}{3}=\frac{2\sqrt{6}}{3}$.

Closest point on a plane to the origin

Method 2: Critical points

2nd derivative test

$$h(x,y) = x^2 + y^2 + (4 - x - 2y)^2$$

$$h_x = 4x + 4y - 8$$
$$h_y = 4x + 10y - 16$$

$$h_{xx} = 4 \quad h_{yy} = 10 \quad h_{xy} = 4$$

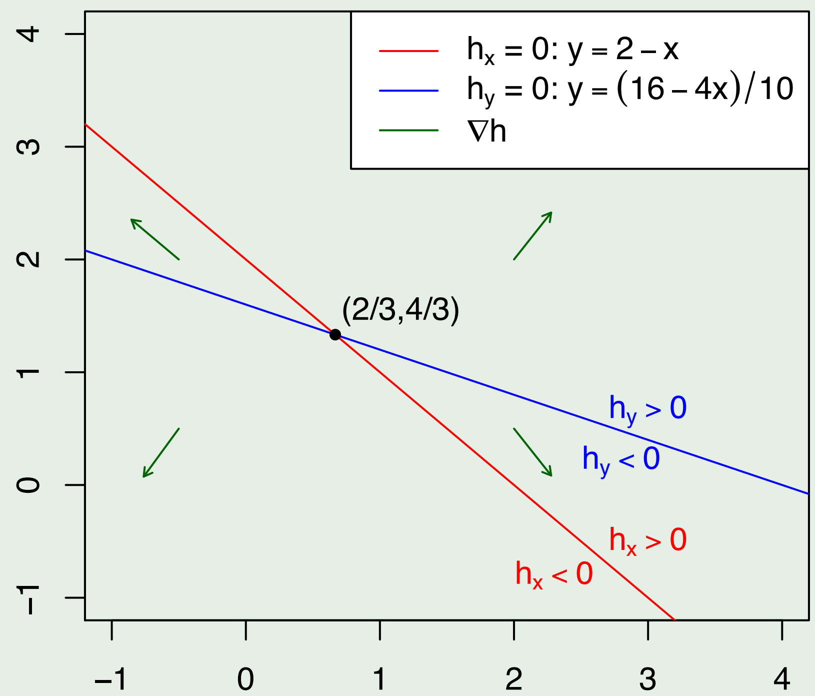
$$D = (4)(10) - 4^2 = 24$$

Since $D > 0$ and $h_{xx} > 0$, it's a local minimum.

Gradient diagram

The plane is split into four regions, according to the signs of h_x and h_y.

h increases as we move away from $\left(\frac{2}{3}, \frac{4}{3}\right)$, so it's an absolute minimum.



Closest point on a plane to the origin

Method 3: Lagrange Multipliers

What point on the plane $z = 4 - x - 2y$ is closest to the origin?

- Rewrite this as a constraint function = constant: $x + 2y + z = 4$
- Minimize $\quad f(x, y, z) = x^2 + y^2 + z^2 \quad$ (square of distance to origin)
  Subject to $\quad g(x, y, z) = x + 2y + z = 4 \quad$ (constraint: on plane)
- Solve $\nabla f = \lambda \nabla g$ and $x + 2y + z = 4$:

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 2, 1 \rangle \qquad x + 2y + z = 4$$

$$2x = \lambda \cdot 1 \quad 2y = \lambda \cdot 2 \quad 2z = \lambda \cdot 1$$

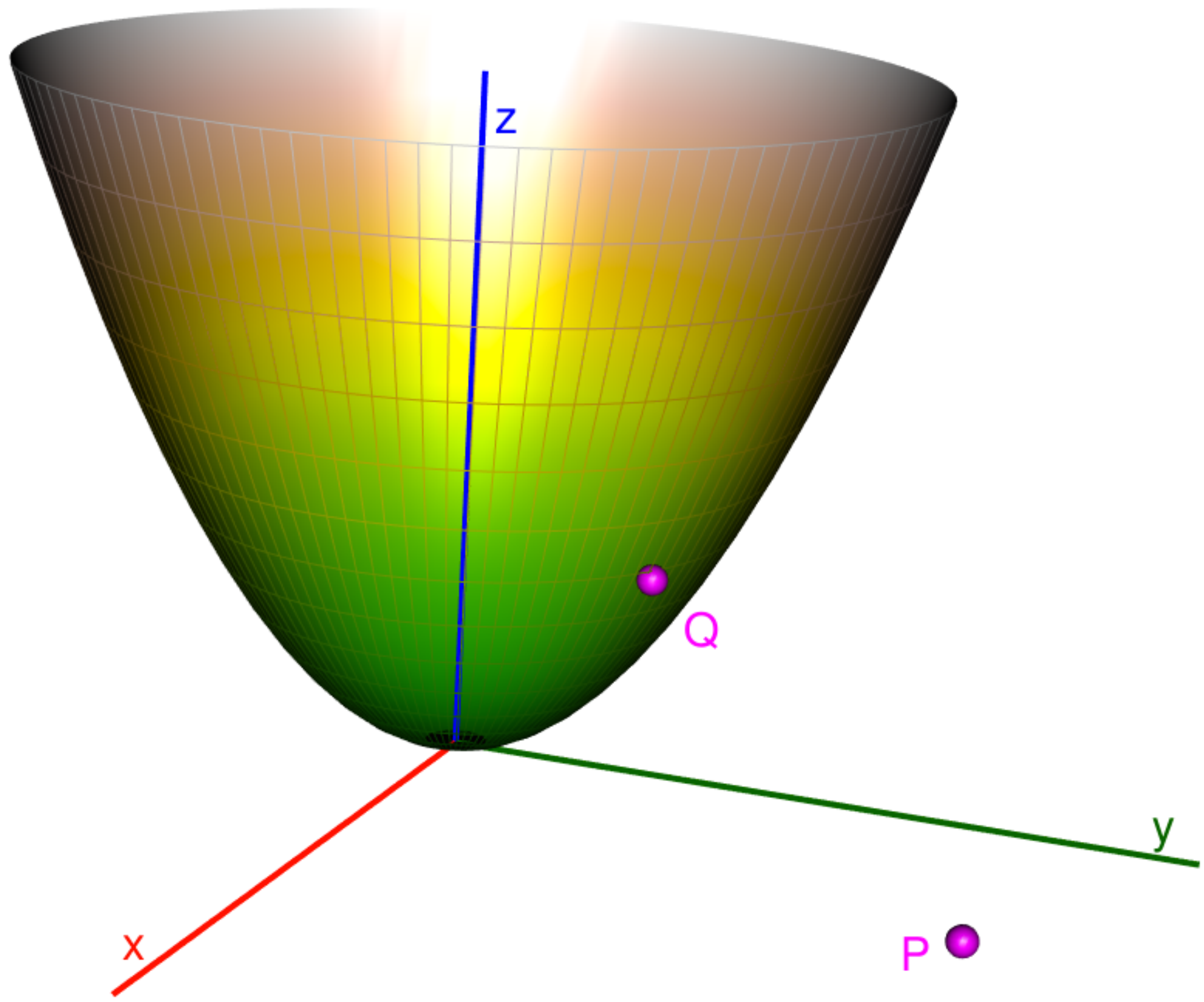
$$x = \tfrac{\lambda}{2} \qquad y = \lambda \qquad z = \tfrac{\lambda}{2} \qquad \tfrac{\lambda}{2} + 2\lambda + \tfrac{\lambda}{2} = 3\lambda = 4 \text{ so } \lambda = \tfrac{4}{3}$$

$$x = \tfrac{2}{3} \qquad y = \tfrac{4}{3} \qquad z = \tfrac{2}{3}$$

- The closest point is $\left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}\right)$.
  Its distance to the origin is $\sqrt{(\frac{2}{3})^2 + (\frac{4}{3})^2 + (\frac{2}{3})^2} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3}$.

Closest point on a surface to a given point

What point Q on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?



Closest point on a surface to a given point

What point Q on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?

- Minimize the square of the distance of P to $Q = (x, y, z)$

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-0)^2$$

subject to the constraint

$$g(x, y, z) = x^2 + y^2 - z = 0$$

- $\nabla f = \langle 2(x-1), 2(y-2), 2z \rangle \qquad \nabla g = \langle 2x, 2y, -1 \rangle$
- Solve $\nabla f = \lambda \nabla g$ and $g(x, y, z) = 0$ for x, y, z, λ:

$$2(x-1) = \lambda(2x) \qquad 2(y-2) = \lambda(2y) \qquad 2z = -\lambda$$
$$x^2 + y^2 - z = 0$$

- Note $x \neq 0$ since the 1st equation would be $-2 = 0$. Similarly, $y \neq 0$. So we may divide by x and y.
- The first three give $\quad \lambda = 1 - \frac{1}{x} = 1 - \frac{2}{y} = -2z \qquad$ so $y = 2x$
- Constraint gives $\quad z = x^2 + y^2 = x^2 + (2x)^2 = 5x^2$

Closest point on a surface to a given point

What point Q on the paraboloid $z = x^2 + y^2$ is closest to $P = (1, 2, 0)$?

- So far, $y = 2x$, $z = 5x^2$, and $\lambda = 1 - \frac{1}{x} = 1 - \frac{2}{y} = -2z$.
- Then $1 - \frac{1}{x} = -2z = -2(5x^2)$ gives $1 - \frac{1}{x} = -10x^2$, so

$$10x^3 + x - 1 = 0$$

- Solve exactly with the cubic equation or approximately with a numerical root finder.

  https://en.wikipedia.org/wiki/Cubic_function#Roots_of_a_cubic_function

  It has one real root (and two complex roots, which we discard):

$$x = \frac{\alpha}{30} - \frac{1}{\alpha} \approx 0.3930027 \qquad \text{where } \alpha = \sqrt[3]{1350 + 30\sqrt{2055}}$$

$$y = 2x \approx 0.7860055 \qquad z = 5x^2 \approx 0.7722557$$

$$Q = (x, 2x, 5x^2) \approx (0.3930027, 0.7860055, 0.7722557)$$