Introduction to Multiple Integrals
Chapters 5.1–5.2 and parts of 5.3–5.5

Prof. Tesler

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Indefinite integrals with multiple variables

- Consider

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$$

- In the input,
  - dx says x is the integration variable.
  - a is constant.
- In the result, C is a constant (does not depend on x).
- Applying d/dx to the result gives back the integrand:

$$\frac{d}{dx}\left(\frac{e^{ax}}{a} + C\right) = e^{ax}$$

Indefinite integrals with multiple variables

- Let x, y, z be variables, and consider

$$\int (2xy + z)\, dz = 2xyz + \frac{z^2}{2} + C(x, y)$$

- In the input:
  - dz says z is the integration variable.
  - x, y are treated as constants while doing the integral.
- In the result:
  - The integration “constant” does not depend on the integration variable z, but it might depend on the other variables x, y! So it’s a function, $C(x, y)$.
- Applying $\partial/\partial z$ to the result gives back the integrand:

$$\frac{\partial}{\partial z}\left(2xyz + \frac{z^2}{2} + C(x, y)\right) = 2xy + z$$

- Note $\frac{\partial}{\partial z} C(x, y) = 0$ for all functions of x and y.

Definite integrals with multiple variables

$$\int_a^b (2xy + z)\,dz = \left(2xyz + \frac{z^2}{2}\right)\Big|_{z=a}^{z=b}$$

As a definite integral:

- The limits a, b may depend on the other variables, x and y.
- Specify limits as $z = a$ and $z = b$ instead of just a and b:

  Don't do this: $\left(2xyz + \frac{z^2}{2}\right)\Big|_a^b$

  This is ambiguous; it doesn't say which of x, y, or z is the variable to set equal to a and to b.
- No need for the integration constant; it will cancel upon subtracting the antiderivatives at the two limits.

Definite integrals with multiple variables

Method 1: Antiderivative at upper limit minus at lower limit

$$\int_0^{x+y} (2xy+z)\,dz = \left(2xyz + \frac{z^2}{2}\right)\Bigg|_{z=0}^{z=x+y}$$

$$= \left(2xy(x+y) + \frac{(x+y)^2}{2}\right) - \left(2xy(0) + \frac{0^2}{2}\right)$$

$$= 2xy(x+y) + \frac{(x+y)^2}{2}$$

Method 2: Subtract term-by-term

$$\int_0^{x+y} (2xy+z)\,dz = \left(2xyz + \frac{z^2}{2}\right)\Bigg|_{z=0}^{z=x+y}$$

$$= 2xy((x+y)-0) + \frac{(x+y)^2 - 0^2}{2}$$

$$= 2xy(x+y) + \frac{(x+y)^2}{2}$$

Iterated integrals

This is a *triple integral*:

$$\int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx$$

- Group it like this, with parentheses:

$$\int_0^1 \left(\int_x^{2x} \left(\int_0^{x+y} 2xy\,dz \right) dy \right) dx$$

- Match up integral signs $\int$ and differentials (like dx) inside-to-out, not left-to-right:
  - **Inside integral:** z goes from 0 to $x+y$
  - **Middle integral:** y goes from x to $2x$
  - **Outside integral:** x goes from 0 to 1.
  - The limits for each variable can only depend on variables that are farther outside than they are:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Iterated integrals

$$I = \int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx$$

- Evaluate the inside integral:

$$\int_0^{x+y} 2xy\,dz = (2xyz)\Big|_{z=0}^{z=x+y} = 2xy((x+y)-0) = 2xy(x+y)$$

- Replace the inside integral by what it evaluates to:

$$I = \int_0^1 \int_x^{2x} 2xy(x+y)\,dy\,dx$$

- Now it's a double integral.
- Iterate! There's a new inside integral; repeat this until all integrals are evaluated.

Iterated integrals

$$I = \int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx = \int_0^1 \int_x^{2x} 2xy(x+y)\,dy\,dx$$

- Iterate! The new inside integral is:

$$\begin{aligned}
\int_x^{2x} 2xy(x+y)\,dy &= \int_x^{2x} (2x^2y + 2xy^2)\,dy \\
&= \left(x^2y^2 + \frac{2xy^3}{3} \right) \Bigg|_{y=x}^{y=2x} \\
&= x^2((2x)^2 - x^2) + \frac{2x((2x)^3 - x^3)}{3} \\
&= x^2(3x^2) + \frac{2x(7x^3)}{3} = 3x^4 + \frac{14x^4}{3} = \frac{23x^4}{3}
\end{aligned}$$

- Replace the inside integral by its evaluation: $I = \int_0^1 \frac{23x^4}{3}\,dx$

Iterated integrals

$$I = \int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx = \cdots = \int_0^1 \frac{23x^4}{3}\,dx$$

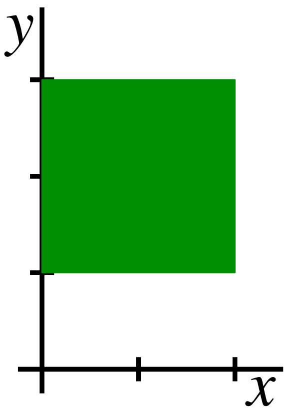
- Now it's down to a single integral.
- Finally,

$$I = \left.\frac{23x^5}{15}\right|_{x=0}^{x=1} = \frac{23(1^5 - 0^5)}{15} = \boxed{\mathbf{\frac{23}{15}}}$$

- Going back to the original problem:

$$\int_0^1 \int_x^{2x} \int_0^{x+y} 2xy\,dz\,dy\,dx = \boxed{\mathbf{\frac{23}{15}}}$$

Rectangle $D = [0,2] \times [1,3]$



- $D = [0,2] \times [1,3]$ is the filled-in rectangle in the xy plane with
$$0 \leqslant x \leqslant 2 \text{ and } 1 \leqslant y \leqslant 3.$$

- Our book often uses R for rectangle and D for any 2-dimensional shape.

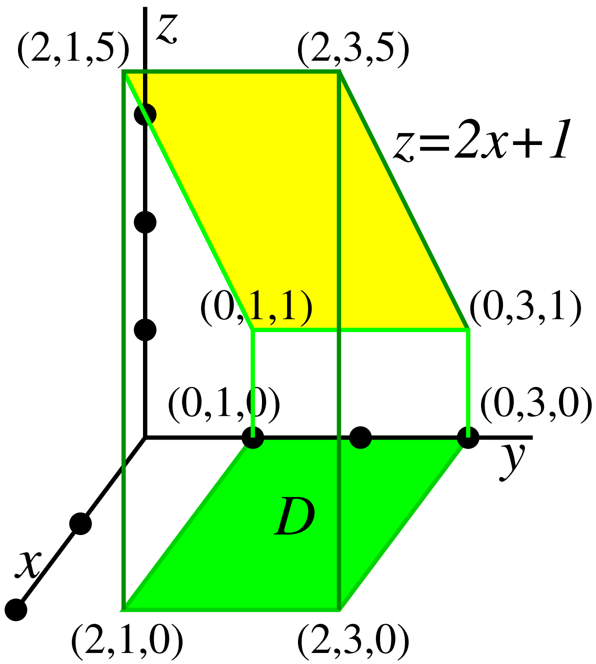
- This is called the *Cartesian product*. In set notation:
$$\begin{aligned} D &= [0,2] \times [1,3] \\ &= \{\,(x,y)\ :\ 0 \leqslant x \leqslant 2 \text{ and } 1 \leqslant y \leqslant 3\,\} \\ &= \{\,(x,y) \mid 0 \leqslant x \leqslant 2 \text{ and } 1 \leqslant y \leqslant 3\,\} \end{aligned}$$
In set notation, some books use a colon : and others use a bar |

- $B = [a,b] \times [c,d] \times [e,f]$ is a filled-in box in 3D:
$$B = \{\,(x,y,z)\ :\ a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d, \quad \text{and} \quad e \leqslant z \leqslant f\,\}.$$

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

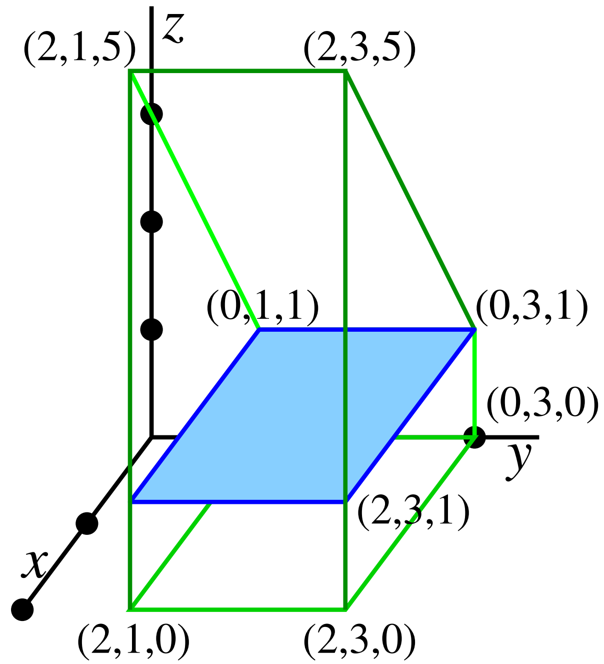
The coordinates on the plane $z = 2x + 1$ above the corners of the rectangle D are

(x, y)	z
$(0, 1)$	1
$(0, 3)$	1
$(2, 1)$	5
$(2, 3)$	5



Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: split into pieces with known volumes



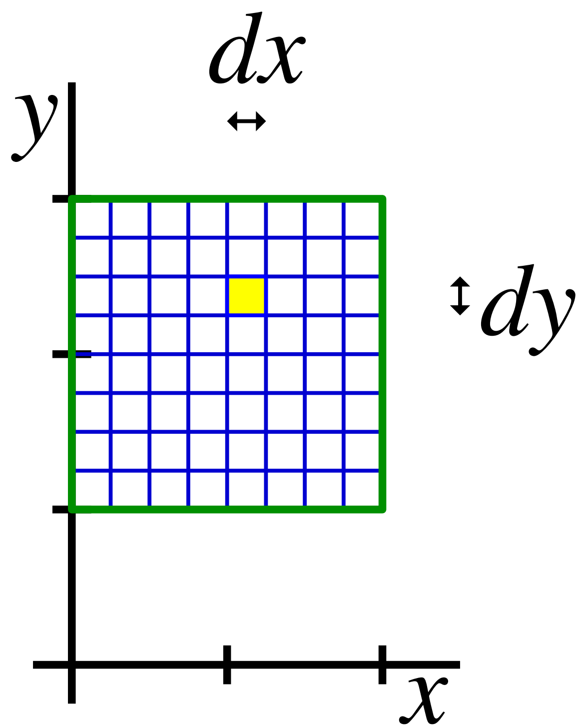
The plane $z = 1$ splits the volume into two parts:

Bottom:	box	$2 \cdot 2 \cdot 1 = 4$
Top:	half a box	$(2 \cdot 2 \cdot 4)/2 = 8$
Total:		12

If x, y, z are in cm, this is 12 cm^3.

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: $dA = dy\,dx$



- Split up D by making a grid with closely spaced horizontal lines and closely spaced vertical lines.
- dA is differential area. It can be $dA = dx\,dy$ or $dA = dy\,dx$.

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: $dA = dy\,dx$

- Let D be a region in the xy plane and let $f(x, y) \geqslant 0$ on D.
- Volume above patch at (x, y) and below $z = f(x, y)$ is

$$\text{(height)(differential area)} = f(x, y)\,dA$$

- $\displaystyle\iint_D f(x, y)\,dA$ is the volume above D and below $z = f(x, y)$.
- For our current example,

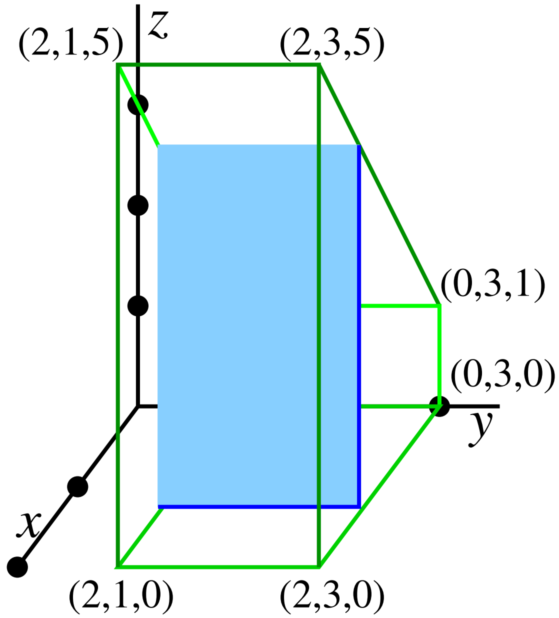
$$\iint_D (2x + 1)\,dA = \boxed{\mathbf{12}}$$

- Use parentheses around $2x + 1$, since it's multiplied by dA.

Do not write it as $\displaystyle\iint_D 2x + 1dA$

Volume under $z = f(x, y)$ and above region D

Cavalieri's Principle



- Let E be a 3D region.
- Let $a \leqslant x \leqslant b$ be the range of x in E.
- Slice E at many values of x;
  e.g., set $\Delta x = (b-a)/n$,
  slice E at $x = a, a + \Delta x, a + 2\Delta x, \ldots, b$,
  and let $n \to \infty$.

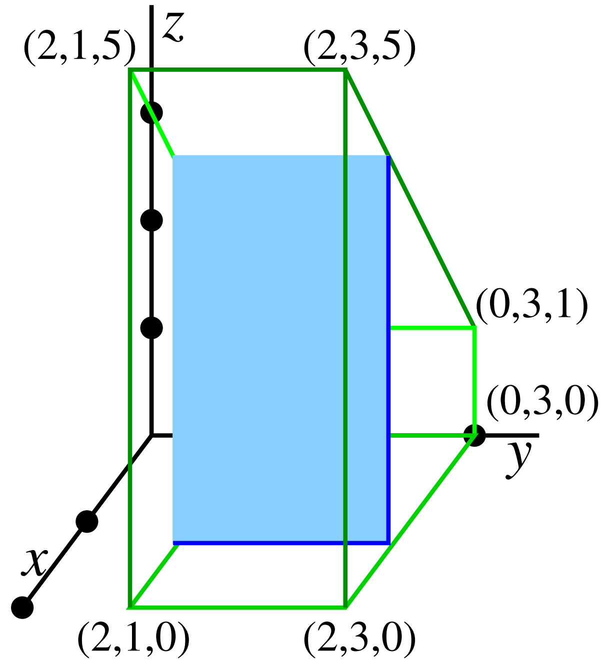
- The infinitesimal cross-section at $x = x_0$ (called an *x-slice*) has area $A(x_0)$, thickness dx, and volume $A(x_0)\,dx$.
- The total volume of E is

$$V = \iint_D f(x,y)\,dA = \int_a^b \text{(area of cross-section at } x)\,dx = \int_a^b A(x)\,dx$$

- This can also be done with y or z cross-sections.

Volume under $z = 2x + 1$ and above rectangle $D = [0,2] \times [1,3]$

Method: $dA = dy\,dx$



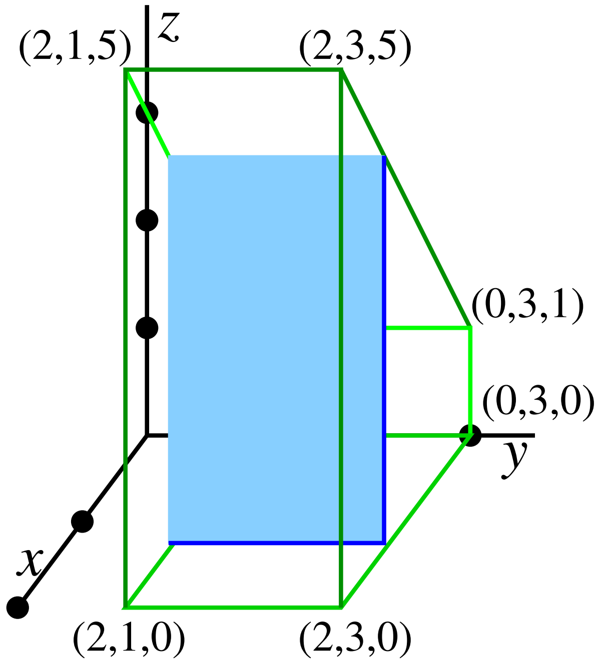
- First, $dA = dy\,dx$. Set up the integral with x, y limits coming from D:

$$\iint_D (2x+1)\,dA = \int_0^2 \int_1^3 (2x+1)\,dy\,dx$$

- The slice at x has infinitesimal thickness dx, area $\int_1^3 (2x+1)\,dy$, and volume $\left(\int_1^3 (2x+1)\,dy\right)dx$.

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: $dA = dy\,dx$



$$\iint_D (2x+1)\,dA = \int_0^2 \int_1^3 (2x+1)\,dy\,dx$$

- Inside:

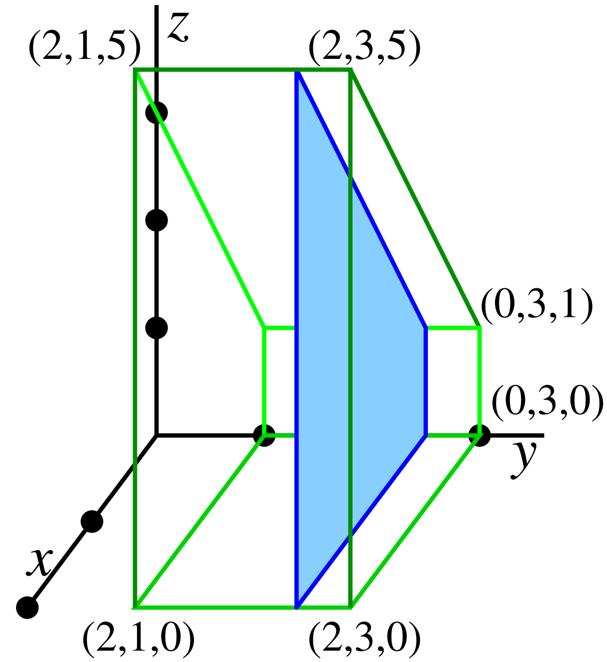
$$\int_1^3 (2x+1)dy = (2x+1)y\Big|_{y=1}^{y=3} = (2x+1)(3-1) = 2(2x+1)$$

- Outside:

$$\int_0^2 2(2x+1)\,dx = 2(x^2+x)\Big|_{x=0}^{x=2} = 2((2^2-0^2)+(2-0))$$
$$= 2(4+2) = \boxed{\mathbf{12}}$$

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: $dA = dx\,dy$



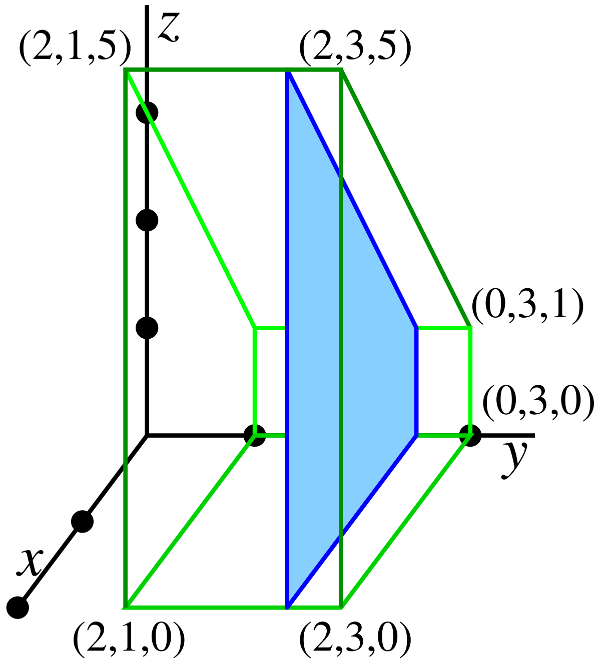
- Next, $dA = dx\,dy$. Set up the integral with x, y limits coming from D:

$$\iint_D (2x + 1)\,dA = \int_1^3 \int_0^2 (2x + 1) dx\,dy$$

- The slice at y has infinitesimal thickness dy, area $\int_0^2 (2x + 1)\,dx$, and volume $(\int_0^2 (2x + 1)\,dx) dy$.

Volume under $z = 2x + 1$ and above rectangle $D = [0,2] \times [1,3]$

Method: $dA = dx\,dy$



$$\iint_D (2x+1)\,dA = \int_1^3 \int_0^2 (2x+1)\,dx\,dy$$

- Inside:

$$\int_0^2 (2x+1)dx = \left.(x^2+x)\right|_{x=0}^{x=2} = (2^2-0^2)+(2-0) = 6$$

- Outside:

$$\int_1^3 6\,dy = \left.6y\right|_{y=1}^{y=3} = 6(3-1) = \boxed{\mathbf{12}}$$

Fubini's Theorem

Let $f(x, y)$ be a continuous function on a rectangle $R = [a, b] \times [c, d]$.
Then

$$\iint_D f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Separation of Variables

- Consider a double integral in this format:

$$\int_a^b \int_c^d f(x)g(y)\,dy\,dx$$

- Since the inside integral is over y, we can factor $f(x)$ out from it:

$$= \int_a^b f(x) \left(\int_c^d g(y)\,dy \right) dx$$

- The y integral has no x, so it's constant for the x integral; factor it out:

$$= \boxed{\left(\int_a^b f(x)\,dx \right) \left(\int_c^d g(y)\,dy \right)}$$

- In the same way,

$$\int_c^d \int_a^b f(x)g(y)\,dx\,dy = \left(\int_a^b f(x)\,dx \right) \left(\int_c^d g(y)\,dy \right)$$

Volume under $z = 2x + 1$ and above rectangle $D = [0, 2] \times [1, 3]$

Method: Separation of Variables

- $2x + 1$ factors as $(2x + 1) \cdot (1)$, so

$$\int_0^2 \int_1^3 (2x+1)\,dy\,dx = \left(\int_0^2 (2x+1)dx\right)\left(\int_1^3 dy\right) = 6 \cdot 2 = \boxed{\mathbf{12}}$$

since

$$\int_0^2 (2x+1)dx = \left.(x^2 + x)\right|_{x=0}^{x=2} = (2^2 - 0^2) + (2 - 0) = 6$$

$$\int_1^3 dy = \left.y\right|_{y=1}^{y=3} = 3 - 1 = 2.$$

Average height

- The average of numbers $x_1, x_2, \ldots, x_n$ is
$$\mu = \frac{x_1 + \cdots + x_n}{n}$$
and it satisfies
$$\underbrace{x_1 + \cdots + x_n}_{n \text{ terms}} = n\mu = \underbrace{\mu + \cdots + \mu}_{n \text{ terms}}$$
- In the same way, the average μ of $f(x, y)$ on region D satisfies
$$\iint\limits_D f(x, y)\, dA = \iint\limits_D \mu\, dA = \mu \iint\limits_D dA$$
so
$$\mu = \frac{\iint_D f(x, y)\, dA}{\iint_D dA} = \frac{\text{Volume between } D \text{ and } z = f(x, y)}{\text{Area of } D}$$
- $\iint_D dA$ sums up differential area patches over D, giving the total area of D.

Average height

In our example, the average of $2x+1$ over $D = [0,2] \times [1,3]$ is

$$\frac{\displaystyle\iint_D (2x+1)\,dA}{\text{Area}(D)} = \frac{12\text{ cm}^3}{4\text{ cm}^2} = \boxed{\textbf{3 cm}}.$$

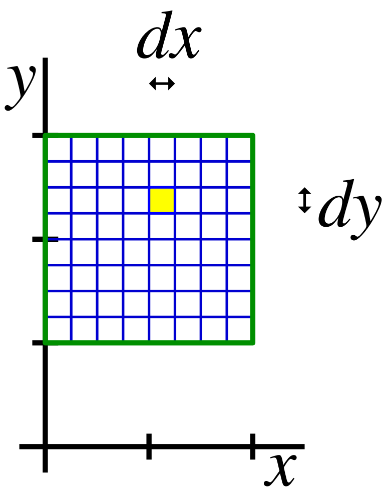
Mean Value Theorem

Mean Value Theorem

If f is continuous on region D and if D is bounded, closed, and connected, then there is a point P in D with $f(P) = \mu$ (the mean value).

- The *mean value* is a technical term for the average value.
- In our example $f(x, y) = 2x + 1$ over the rectangle $D = [0, 2] \times [1, 3]$, the mean is $\mu = 3$.
- Solving $f(x, y) = \mu$ gives $2x + 1 = 3$, so $x = (3 - 1)/2 = 1$.
- Within D, all points on the line segment $x = 1$ and $1 \leqslant y \leqslant 3$ give $f(1, y) = 3$.

Density



- We spread butter on a piece of bread, $D = [0, 2] \times [1, 3]$.
- It's spread unevenly, giving varying density.
- **Units:** x, y: cm mass: g density: g/cm^2

- **Density at (x, y):** $\rho(x, y) = 2x + 1$
- **Mass of the tiny patch at (x, y):**
$$\rho(x, y)\, dA = (2x + 1)\, dA$$
- **Total mass of D:**
$$\iint_D \rho(x, y)\, dA = \iint_D (2x + 1)\, dA = 12 \text{ g}$$

$$\textbf{Average density} = \frac{\text{total mass}}{\text{total area}} = \frac{\iint_D \rho(x, y)\, dA}{\iint_D dA} = \frac{12 \text{ g}}{4 \text{ cm}^2} = 3 \text{ g/cm}^2$$