3.1 Iterated Partial Derivatives

Prof. Tesler

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Higher Derivatives

- Take the partial derivative of $f(x, y) = x^2y^3$ with respect to x:

$$f_x(x, y) = 2xy^3$$

- This is also a function of x and y, and we can take another derivative with respect to either variable:
  - The x derivative of $f_x(x, y)$ is $(f_x)_x = f_{xx} = 2y^3$.
  - The y derivative of $f_x(x, y)$ is $(f_x)_y = f_{xy} = 6xy^2$.
  - f_{xx} and f_{xy} are each an *iterated partial derivative of second order*.

- The y derivative of the x derivative can also be written:

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}(x^2y^3) = \frac{\partial}{\partial y}(2xy^3) = 6xy^2 \qquad \text{or} \qquad \frac{\partial^2}{\partial y\,\partial x}(x^2y^3) = 6xy^2$$

Iterated Derivative Notations

- Let $f(x, y) = x^2y^3$.
- There are two notations for partial derivatives, f_x and $\frac{\partial f}{\partial x}$.

Partial derivative of f with respect to x in each notation:

$$f_x = 2xy^3 \qquad \frac{\partial}{\partial x} f(x, y) = \frac{\partial f}{\partial x} = 2xy^3$$

Partial derivative of that with respect to y:

$(f_x)_y = f_{xy}$,
so $f_{xy}(x, y) = 6xy^2$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f\right) = \frac{\partial^2}{\partial y\, \partial x} f = \frac{\partial^2 f}{\partial y\, \partial x} = 6xy^2$$

- Notice derivatives are listed in the opposite order in each notation.

Iterated Derivative Notations

- Notice derivatives are listed in the opposite order in each notation.
- In each notation, compute the derivatives in order from the one listed closest to f, to the one farthest from f:

$$f_{xyzzy} = \frac{\partial}{\partial y}\frac{\partial}{\partial z}\frac{\partial}{\partial z}\frac{\partial}{\partial y}\frac{\partial}{\partial x} f = \frac{\partial^5}{\partial y\,\partial z^2\,\partial y\,\partial x} f = \frac{\partial^5 f}{\partial y\,\partial z^2\,\partial y\,\partial x}$$

- Both notations say to take derivatives in the order x, y, z, z, y.

Mixed Partial Derivatives

$$f(x,y) = x^2y^3 \qquad f_x = 2xy^3 \qquad f_y = 3x^2y^2$$

$$f_{xx} = 2y^3 \qquad f_{yx} = 6xy^2$$

$$f_{xy} = 6xy^2 \qquad f_{yy} = 6x^2y$$

- A *mixed partial derivative* has derivatives with respect to two or more variables.
- f_{xy} and f_{yx} are mixed.
  f_{xx} and f_{yy} are not mixed.
- In this example, notice that $f_{xy} = f_{yx} = 6xy^2$.
  The order of the derivatives did not affect the result.

Continuity Notation

- A function $f(x, y, z)$ is in *class C^1* if f and its first derivatives f_x, f_y, f_z are defined and continuous.
- *Class C^2:* f and all of its first derivatives (f_x, f_y, f_z) and second derivatives $(f_{xx}, f_{xy}, f_{xz}, f_{yx}, f_{yy}, f_{yz}, f_{zx}, f_{zy}, f_{zz})$ are defined and continuous.
- *Class C^n:* f and all of its first, second, ..., n^{th} derivatives are defined and continuous.

Clairaut's Theorem

If f and its first and second derivatives are defined and continuous (that is, f is class C^2), then $f_{xy} = f_{yx}$.

Example

The function $f(x,y) = x^2y^3$ is C^2 (in fact, C^∞), and $f_{xy} = f_{yx} = 6xy^2$.

Example with higher order derivatives

- As long as f and all its derivatives are defined and continuous up to the required order, you can change the order of the derivatives.
- E.g., if $f(x,y,z)$ is class C^5, then

$$f_{xyzzy} = f_{xyyzz} \qquad \frac{\partial^5 f}{\partial y\,\partial z^2\,\partial y\,\partial x} = \frac{\partial^5 f}{\partial x\,\partial y^2\,\partial z^2}$$

Clairaut's Theorem

Example

- Is there a C^2 function $f(x,y)$ with $f_x = \cos(x+y)$ and $f_y = \ln(x+y)$?
- If so, Clairaut's Theorem says $f_{xy} = f_{yx}$.
- $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\cos(x+y) = -\sin(x+y)$
- $f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\ln(x+y) = \frac{1}{x+y}$
- These don't agree, so there is no such function.

Example

The book, p. 158, 3.1#32, has an example where $f_{xy}(a,b) \neq f_{yx}(a,b)$ at a point (a,b) that has discontinuous 2$^{\text{nd}}$ derivatives.

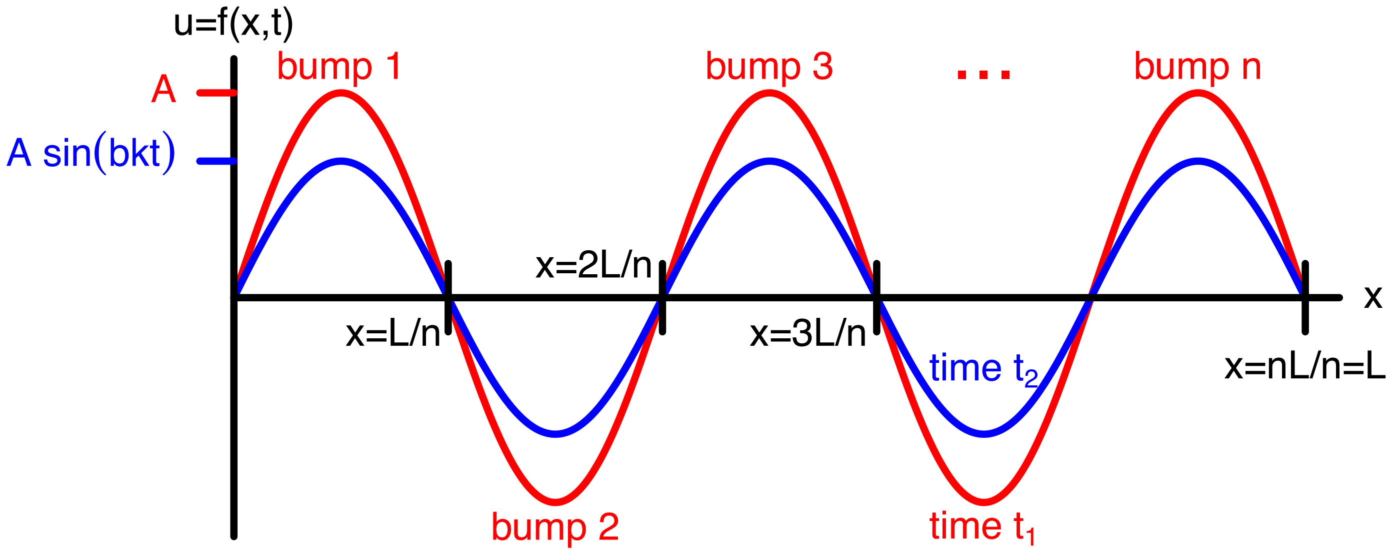
Differential Equations

- Many laws of nature in Physics and Chemistry are expressed using *differential equations*.
- *Ordinary Differential Equations (ODEs):*
  You're given an equation in x, y and derivatives $y', y'', \ldots$.
  The goal is to find a function $y = f(x)$ satisfying the equation.
- ODEs were introduced in Math 20B.
  Solution methods will be covered in Math 20D.
  For now, we will just show how to verify a solution.

Example: Solve $y' = 2y$

- The answer turns out to be $y = Ce^{2x}$, where C is any constant.
- Verify this is a solution:
  - Left side: $y' = C(2e^{2x}) = 2Ce^{2x}$
  - Right side: $2y = 2Ce^{2x}$.
  - They're equal, so $y' = 2y$.

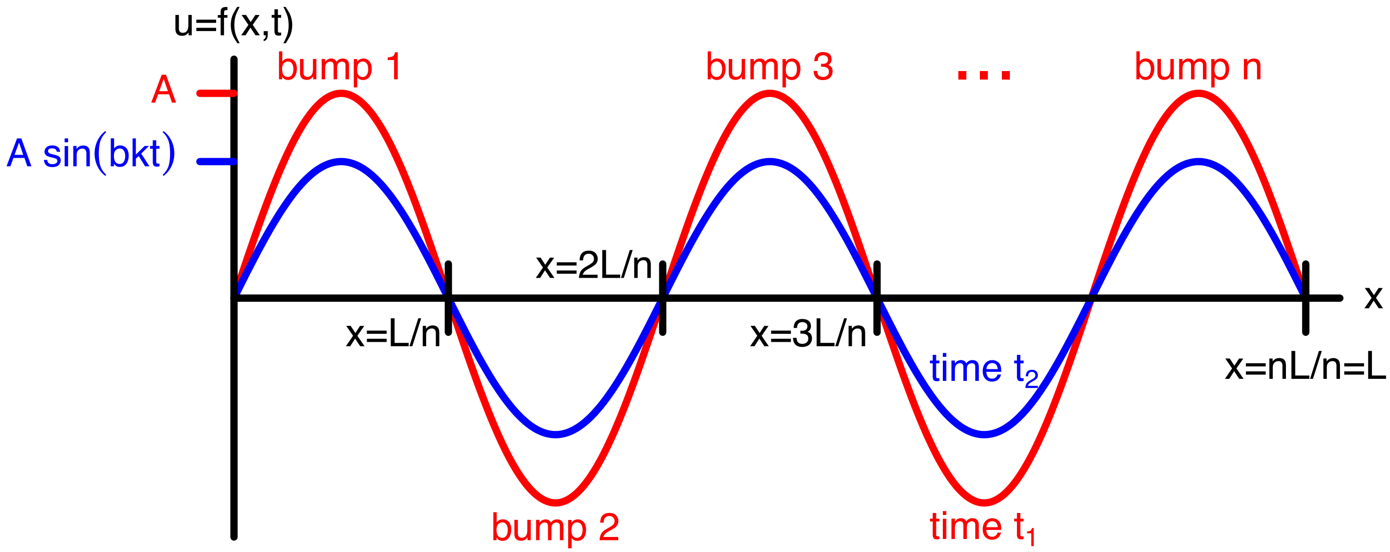
Wave Equation



- A *partial differential equation* (PDE) has a function of multiple variables, and partial derivatives.
- The *wave equation* describes motion of waves.
- We'll study the *one dimensional wave equation:*

$$u_{tt} = b^2 u_{xx} \qquad \text{(where } b \text{ is constant)}$$

- Goal is to solve for a function $u = f(x,t)$ satisfying this equation.



Parameters (constant)	**Variables**
$A =$ amplitude	$x =$ horizontal position
$L =$ length	$t =$ time
$n =$ # bumps	$u = f(x,t) = y$-coordinate
$b =$ oscillation speed	

Wave Equation

$$u_{tt} = b^2\, u_{xx} \qquad \text{or} \qquad \frac{\partial^2 u}{\partial t^2} = b^2\, \frac{\partial^2 u}{\partial x^2}$$

- The solution is

$$u = A \sin(bkt) \sin(kx) \qquad \text{where} \qquad k = n\pi/L.$$

Verify the equation $u_{tt} = b^2 u_{xx}$:

Left side: compute u_{tt}

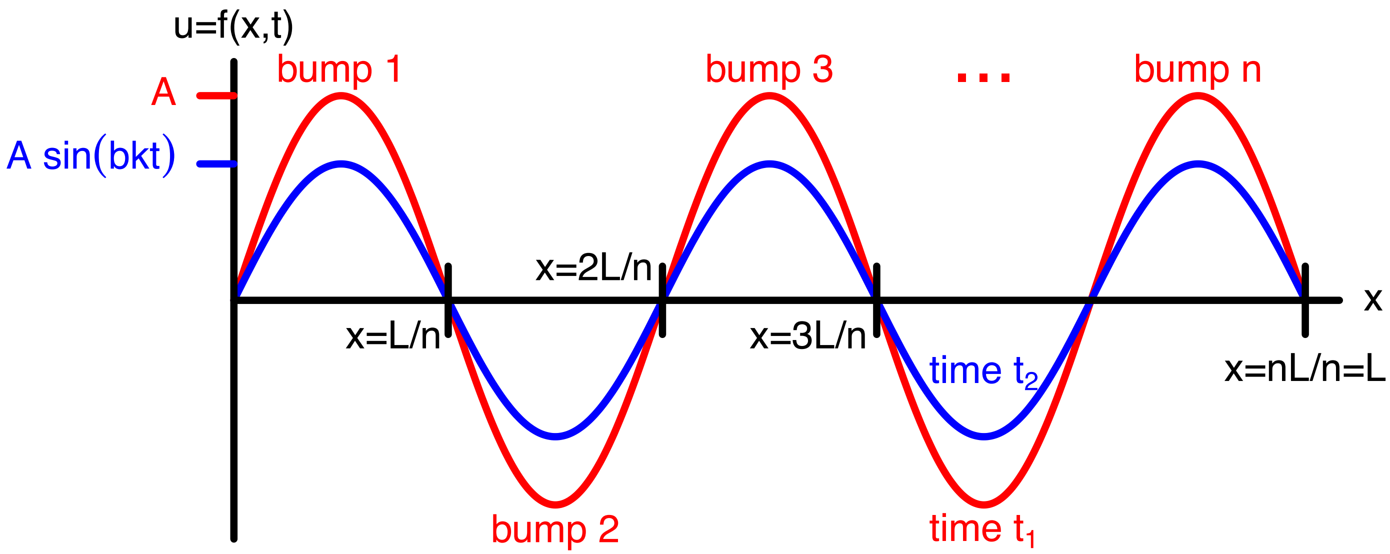
$$u_t = Abk \cos(bkt) \sin(kx)$$
$$u_{tt} = -A(bk)^2 \sin(bkt) \sin(kx)$$

Right side: compute $b^2\, u_{xx}$

$$u_x = Ak \sin(bkt) \cos(kx)$$
$$u_{xx} = -Ak^2 \sin(bkt) \sin(kx)$$
$$b^2\, u_{xx} = -Ak^2 b^2 \sin(bkt) \sin(kx)$$

The left and right sides are equal, so it's a solution.

Wave Equation



- $A\ \sin(bkt)$ is the amplitude at time t.
  This varies between $\pm A$, so the maximum amplitude is A.
- $k = \frac{n\pi}{L}$ so $\sin(kx) = \sin(\frac{n\pi x}{L})$.
  So $\sin(kx) = 0$ at $x = 0, \frac{L}{n}, \frac{2L}{n}, \ldots, \frac{nL}{n}$.
  Hence those points are on the x-axis at all times t.

Implicit Equations and Partial Derivatives

- $z = \sqrt{1 - x^2 - y^2}$ gives $z = f(x, y)$ *explicitly*.
- $x^2 + y^2 + z^2 = 1$ gives z in terms of x and y *implicitly*.
  For each x, y, one can solve for the value(s) of z where it holds.
- $\sin(xyz) = x + 2y + 3z$ cannot be solved explicitly for z.

Implicit Equations and Partial Derivatives

- To compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ with an implicit equation:
  - Assume $z = f(x, y)$.
  - For $\frac{\partial}{\partial x}$, treat x as a variable
    y as a constant
    z as a function of x, y

$x^2 + y^2 + z^2 = 1$. Find $\partial z / \partial x$.

- **Left side:** $\frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x + 0 + 2z\frac{\partial z}{\partial x}$
- **Right side:** $\frac{\partial}{\partial x}(1) = 0$
- **Combine:** $2x + 0 + 2z\frac{\partial z}{\partial x} = 0$
- **Solve:** $\frac{\partial z}{\partial x} = -x/z$

Implicit Equations and Partial Derivatives

$x^2+y^2+z^2=1$. Find $\partial z/\partial x$ at $(x,y)=(1/3,2/3)$.

- At $(x,y)=(1/3,2/3)$:

$$\left(\tfrac{1}{3}\right)^2+\left(\tfrac{2}{3}\right)^2+z^2=1$$

$$z^2=1-\tfrac{1}{9}-\tfrac{4}{9}=\tfrac{4}{9}, \quad \text{so} \quad z=\pm\tfrac{2}{3}$$

- At $(x,y,z)=(1/3,2/3,2/3)$:

$$\frac{\partial z}{\partial x}=-x/z=-\frac{1/3}{2/3}=\boxed{-\frac{\mathbf{1}}{\mathbf{2}}}$$

- At $(x,y,z)=(1/3,2/3,-2/3)$:

$$\frac{\partial z}{\partial x}=-x/z=-\frac{1/3}{-2/3}=\boxed{\frac{\mathbf{1}}{\mathbf{2}}}$$

Implicit Equations and Partial Derivatives

$$\sin(xyz) = x + 2y + 3z$$

- Find $\frac{\partial z}{\partial x}$ in the above equation:

  $\frac{\partial}{\partial x}$ **of left side:** $\cos(xyz) \cdot \left(yz + xy\frac{\partial z}{\partial x}\right)$

  $\frac{\partial}{\partial x}$ **of right side:** $1 + 0 + 3\frac{\partial z}{\partial x}$

  Combined: $\cos(xyz) \cdot \left(yz + xy\frac{\partial z}{\partial x}\right) = 1 + 3\frac{\partial z}{\partial x}$

- Solve this for $\partial z/\partial x$:

$$1 - yz\cos(xyz) = \frac{\partial z}{\partial x}\left(xy\cos(xyz) - 3\right)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3}}$$

Implicit Equations and Partial Derivatives

$$\sin(xyz) = x + 2y + 3z$$

- Find $\partial z/\partial x$ at $(x, y) = (0, 0)$.
- At $(x, y) = (0, 0)$, the equation becomes

$$\sin(0) = 0 + 2(0) + 3z \qquad \text{so } z = 0.$$

- Plug numerical values of x, y, z into the formula for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3} = \frac{1 - 0\cos(0)}{0\cos(0) - 3} = \frac{1}{-3} = \boxed{-\frac{\mathbf{1}}{\mathbf{3}}}$$

Implicit Equations and Partial Derivatives

$$\sin(xyz) = x + 2y + 3z$$

- Find $\partial z/\partial x$ at $(x, y) = (1, -.1)$.
- $\sin((1)(-.1)z) = 1 + 2(-.1) + 3z$, so $\sin(-0.1z) = .8 + 3z$.
- Use a numerical solver to get $z \approx -0.2580654401$.
- Plug $x = 1$, $y = -.1$, $z \approx -0.2580654401$ into formula for $\partial z/\partial x$:

$$\frac{\partial z}{\partial x} = \frac{1 - yz\cos(xyz)}{xy\cos(xyz) - 3} \approx \boxed{\mathbf{-0.3142621009}}$$