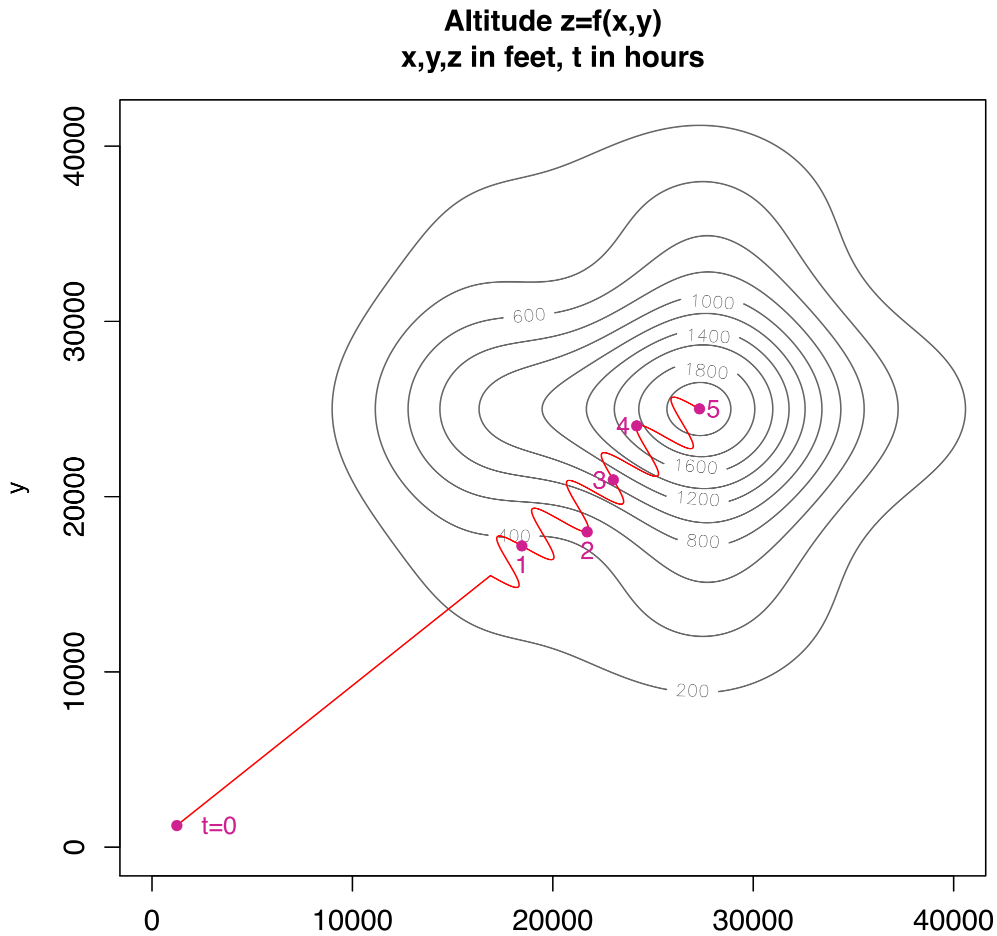
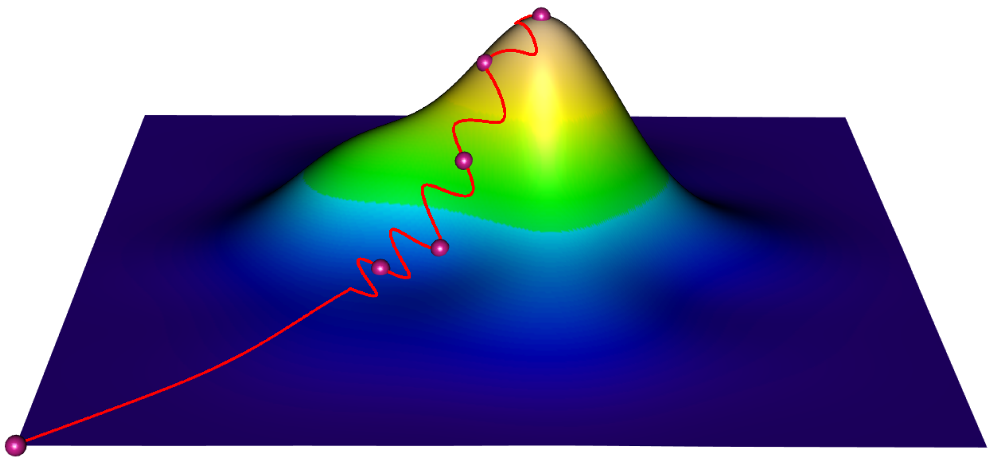
2.6 Gradients and Directional Derivatives

Prof. Tesler

Math 20C
Fall 2018

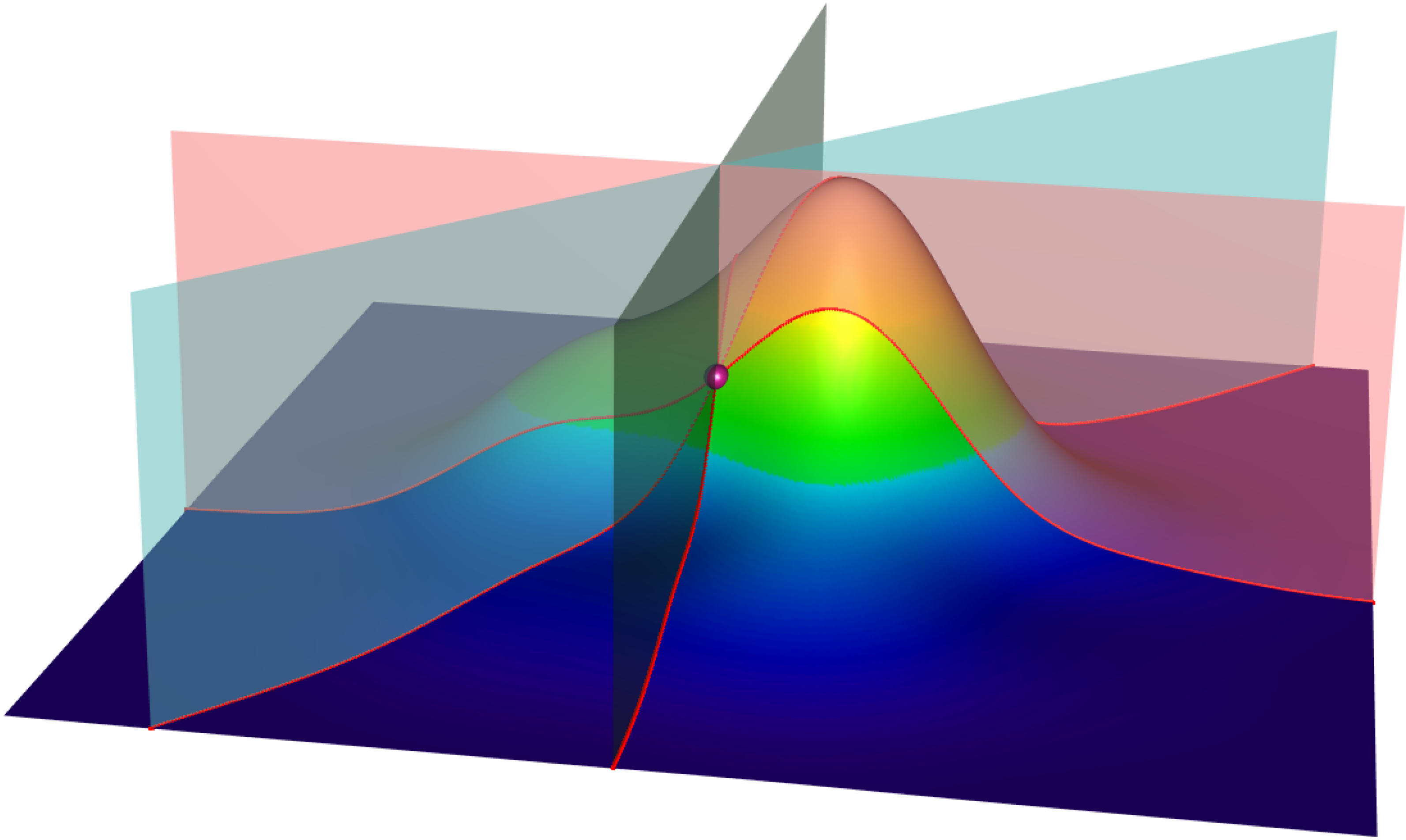
Hiking trail and chain rule



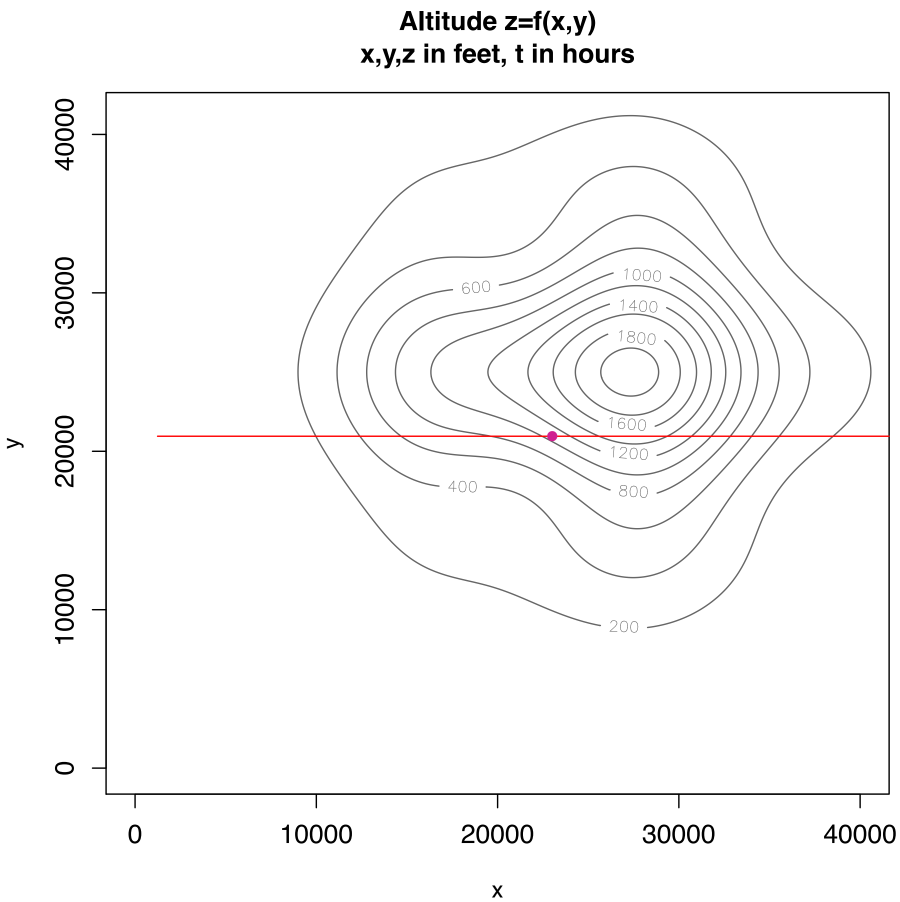


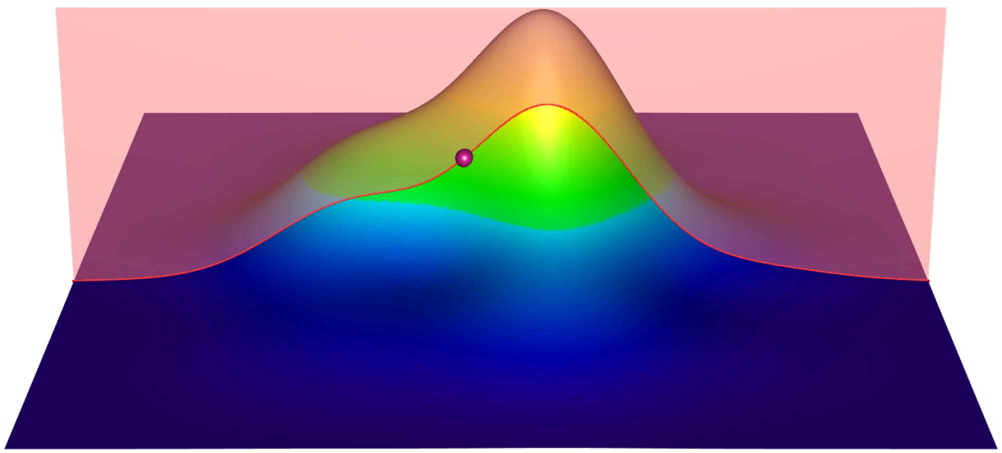
- A mountain has altitude $z = f(x, y)$ above point (x, y).
- Plot a hiking trail $(x(t), y(t))$ on the contour map.
  This gives altitude $z(t) = f(x(t), y(t))$, and 3D trail $(x(t), y(t), z(t))$.
- We studied using the chain rule to compute the hiker's vertical speed, dz/dt.

How steep are different cross-sections of a mountain?



Partial derivatives

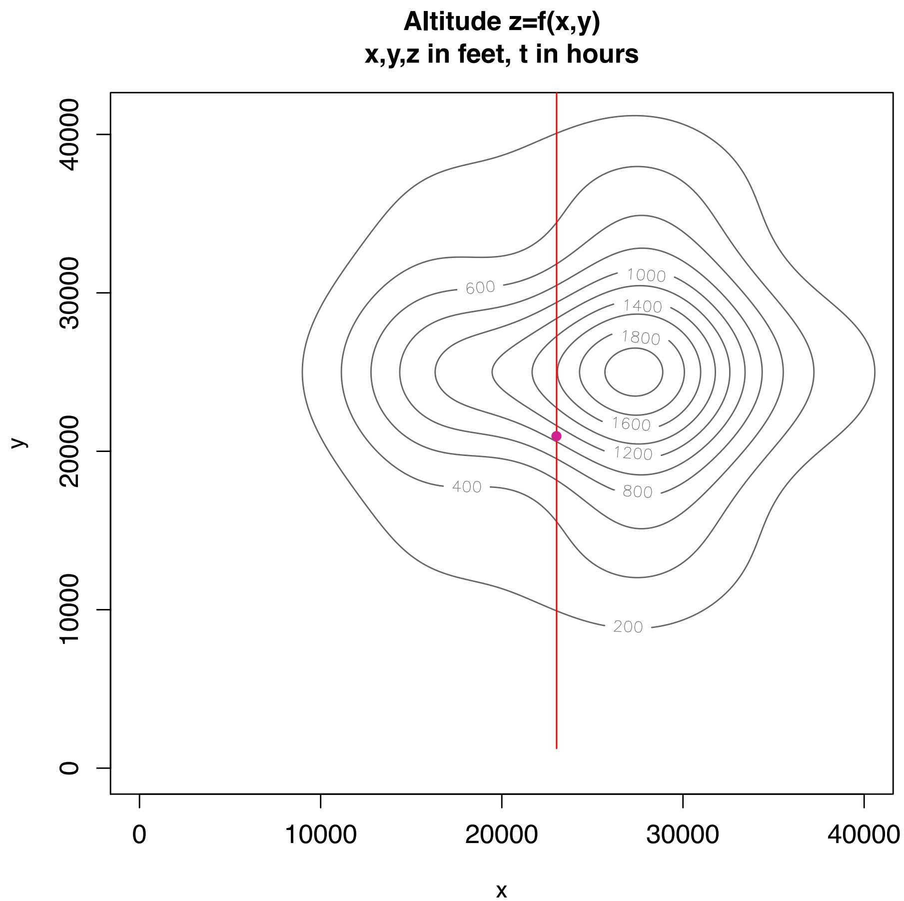


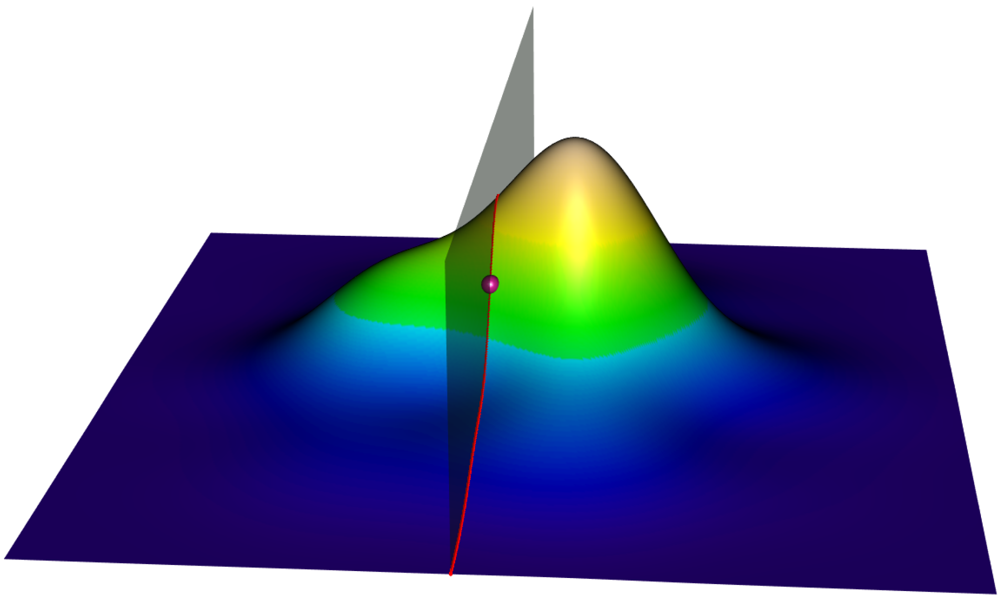


Slope at point $P = (x, y) = (a, b)$ when traveling east $\rightarrow$

- Hold y constant ($y = b$) and vary x, giving $z = f(x, b)$.
- Get a 2D curve in the vertical plane $y = b$.
- Slope at P is $\frac{\partial z}{\partial x} = f_x(a, b)$.

Partial derivatives

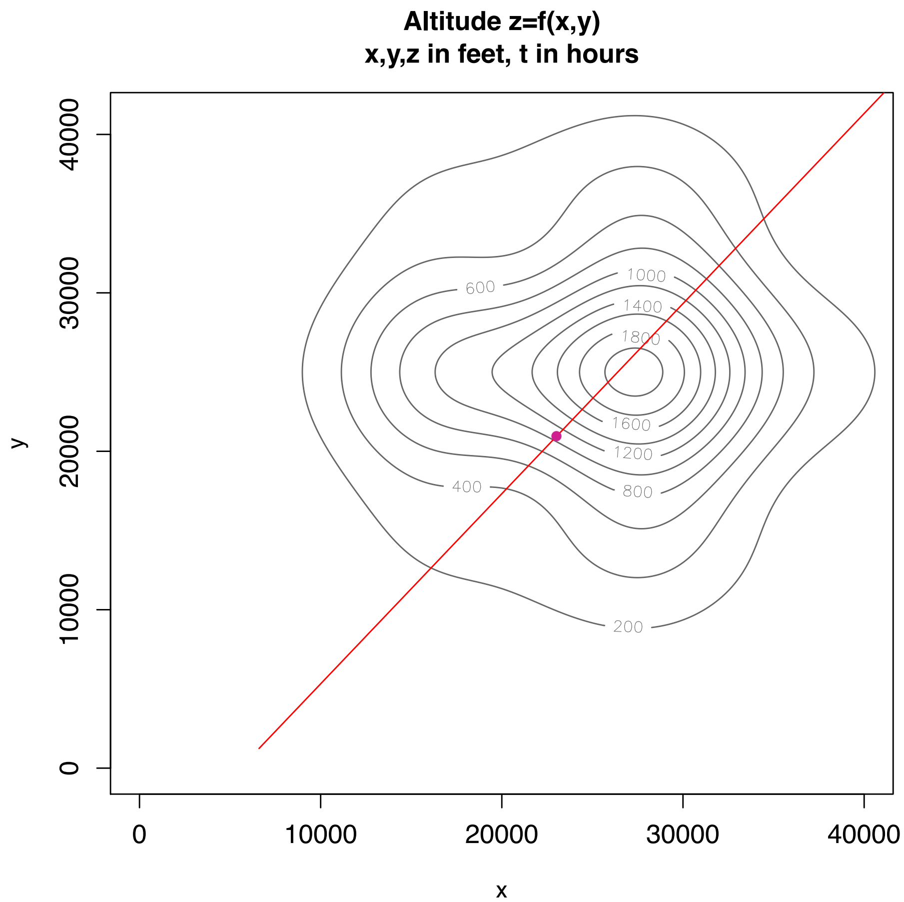


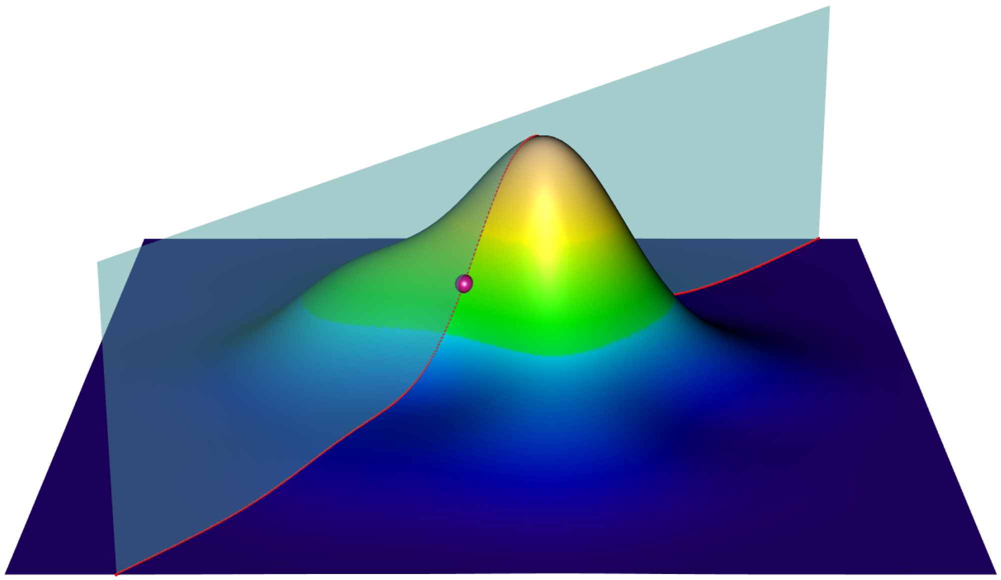


Slope at point $P = (x, y) = (a, b)$ when traveling north $\uparrow$

- Hold x constant ($x = a$) and vary y, giving $z = f(a, y)$.
- Get a 2D curve in the vertical plane $x = a$.
- Slope at P is $\frac{\partial z}{\partial y} = f_y(a, b)$.

Directional derivatives

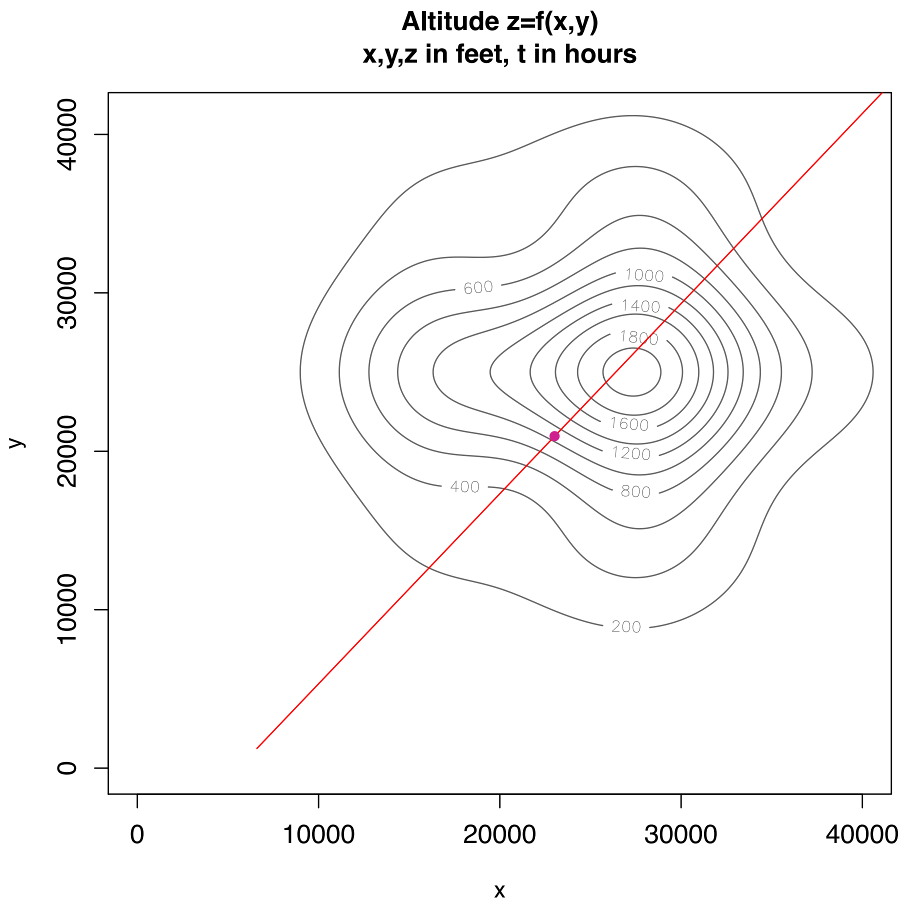




Slope at $P = (x, y) = (a, b)$ when traveling on diagonal line $\nearrow$

- On the 2D contour map, draw a diagonal line through P.
- On the 3D plot, this is a 2D curve on a vertical cross-section.
- What's the slope when traveling through P along this curve?

Directional derivatives



- Let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector in the xy plane.
- On the map, travel on the line through (a, b) with direction $\vec{u}$:

$$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle a, b \rangle + t\,\vec{u} \,.$$

- Each (x, y) point gives a z coordinate via $z = f(x, y)$.

Directional derivatives

- Traveling on line $\vec{r}(t) = \langle x(t), y(t) \rangle = \langle a, b \rangle + t\,\vec{u}$:

Time	(x,y)	z
$t=0$	(a,b)	$f(a,b)$
$t=\Delta t$	$(a+u_1\,\Delta t,\, b+u_2\,\Delta t)$	$f(a+u_1\,\Delta t,\, b+u_2\,\Delta t)$

- Between times 0 and Δt, the change in altitude is

$$\begin{aligned}\Delta z &= f(a+u_1\,\Delta t,\, b+u_2\,\Delta t) - f(a,b)\\ &\approx f_x(a,b)\,u_1\,\Delta t + f_y(a,b)\,u_2\,\Delta t = \nabla f(a,b)\cdot\vec{u}\,\Delta t\end{aligned}$$

- The horizontal change (in the xy plane) is

$$\|\vec{u}\,\Delta t\| = \|\vec{u}\|\;\Delta t = 1\,\Delta t = \Delta t$$

- The slope on the mountain at $(x,y)=(a,b)$ in that cross-section is

$$\frac{\text{Vertical change}}{\text{Horizontal change}} = \frac{\Delta z}{\Delta t} \approx \nabla f(a,b)\cdot\vec{u}$$

- As $\Delta t \to 0$, this gives the instantaneous rate of change:

$$\nabla f(a,b)\cdot\vec{u}$$

Directional derivatives — Second method

- Let $\vec{u} = \langle u_1, u_2 \rangle$ be a unit vector, and travel on line $\vec{r}(t) = \langle a, b \rangle + t\,\vec{u}$.
- Time $t = 0$ corresponds to point $P = (a, b)$.
- Use the chain rule to find the instantaneous slope at time $t = 0$:

$$\begin{aligned}
\frac{d}{dt} f(\vec{r}(t))\Big|_{t=0} &= \frac{d}{dt} f(\langle a, b \rangle + t\vec{u})\Big|_{t=0} \\
&= \left(\nabla f \cdot \vec{r}\,'(t)\right)\Big|_{t=0} \\
&= \nabla f(a, b) \cdot \vec{r}\,'(0) \\
&= \nabla f(a, b) \cdot \vec{u}
\end{aligned}$$

Directional derivatives

The *directional derivative* of $f(\vec{x})$ in the direction $\vec{u}$ (a unit vector) is

$$D_{\vec{u}} f(\vec{x}) = \frac{d}{dt} f(\vec{x} + t\vec{u}) \Big|_{t=0} \qquad \text{(useful theoretically)}$$

$$= \boxed{\nabla f(\vec{x}) \cdot \vec{u}} \qquad \text{(easier for computations)}$$

Notation warning

Df for the derivative matrix and $D_{\vec{u}} f$ for directional derivative are completely different, even though the notations look similar.

Directional derivatives

The *directional derivative* of $f(\vec{x})$ in the direction $\vec{u}$ (a unit vector) is

$$D_{\vec{u}}\, f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}$$

Examples

- $D_{\hat{\imath}}\, f(a,b) = \nabla f(a,b) \cdot \hat{\imath} = \left\langle f_x(a,b), f_y(a,b) \right\rangle \cdot \hat{\imath} = f_x(a,b)$
- $D_{\hat{\jmath}}\, f(a,b) = \nabla f(a,b) \cdot \hat{\jmath} = \left\langle f_x(a,b), f_y(a,b) \right\rangle \cdot \hat{\jmath} = f_y(a,b)$

Be careful: $\vec{u}$ must be a unit vector

- $D_{2\hat{\imath}}\, f(a,b) = \nabla f(a,b) \cdot 2\hat{\imath} = \left\langle f_x(a,b), f_y(a,b) \right\rangle \cdot 2\hat{\imath} = 2 f_x(a,b)$
- $\hat{\imath}$ and $2\hat{\imath}$ have the same direction, but this is not the slope; it's off by a factor of 2.

Example

Find the directional derivative of $f(x,y,z) = x^2 - 3xy + z^3$ at the point $P = (1,2,3)$ in the direction towards $Q = (6,5,4)$.

We'll apply the formula $D_{\vec{u}} f = \vec{u} \cdot \nabla f$.

Gradient

- The gradient (as a function):

$$\nabla f = \left\langle 2x - 3y, -3x, 3z^2 \right\rangle$$

- The gradient at point P:

$$\nabla f(1,2,3) = \left\langle 2(1) - 3(2), -3(1), 3(3^2) \right\rangle = \langle -4, -3, 27 \rangle$$

Example

Find the directional derivative of $f(x,y,z) = x^2 - 3xy + z^3$ at the point $P = (1,2,3)$ in the direction towards $Q = (6,5,4)$.

Direction vector

- The vector from P to Q is

$$\vec{v} = \overrightarrow{PQ} = \langle 5, 3, 1 \rangle$$

- However, this is not a unit vector. It has length

$$\|\vec{v}\| = \sqrt{5^2 + 3^2 + 1^2} = \sqrt{25 + 9 + 1} = \sqrt{35}$$

- Unit vector:

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 5, 3, 1 \rangle}{\sqrt{35}}$$

Example

Find the directional derivative of $f(x,y,z) = x^2 - 3xy + z^3$ at the point $P = (1,2,3)$ in the direction towards $Q = (6,5,4)$.

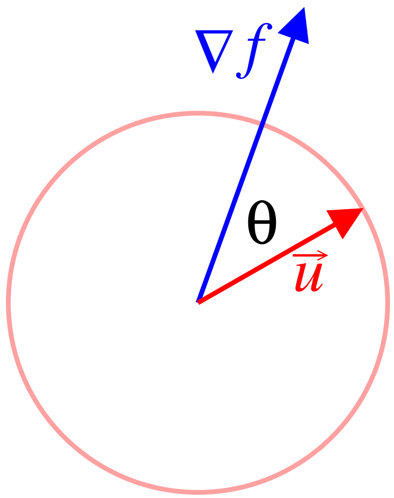
- So far:

$$\nabla f(1,2,3) = \langle -4, -3, 27 \rangle \qquad \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 5,3,1 \rangle}{\sqrt{35}}$$

- The directional derivative at this point:

$$\begin{aligned} D_{\vec{u}} f(1,2,3) &= \vec{u} \cdot \nabla f(1,2,3) \\ &= \frac{\langle 5,3,1 \rangle}{\sqrt{35}} \cdot \langle -4,-3,27 \rangle = \frac{5(-4) + 3(-3) + 1(27)}{\sqrt{35}} \\ &= \boxed{-\frac{\mathbf{2}}{\sqrt{\mathbf{35}}}} \end{aligned}$$

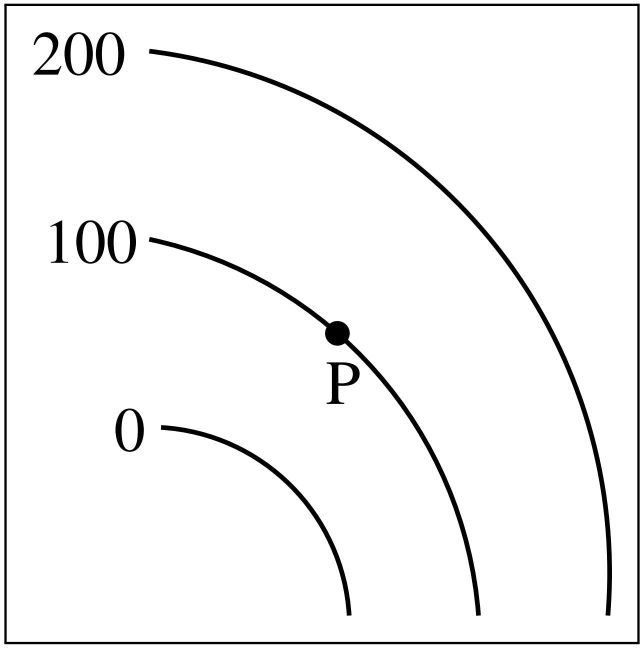
Possible values of $D_{\vec{u}}\,f$



For a function $z = f(x, y)$ and a point $P = (a, b)$, what are the possible values of $D_{\vec{u}}\,f(a, b)$ as $\vec{u}$ varies over all directions?

- $D_{\vec{u}}\,f = \vec{u} \cdot \nabla f = \|\vec{u}\|\ \|\nabla f\|\ \cos(\theta)$
- $\vec{u}$ is a unit vector, so $\|\vec{u}\| = 1$ and $D_{\vec{u}}\,f = \|\nabla f\|\ \cos(\theta)$.
- As $\vec{u}$ varies, $\cos(\theta)$ varies between ± 1.
- So $D_{\vec{u}}\,f$ varies between $\pm\,\|\nabla f\|$.

Special directions

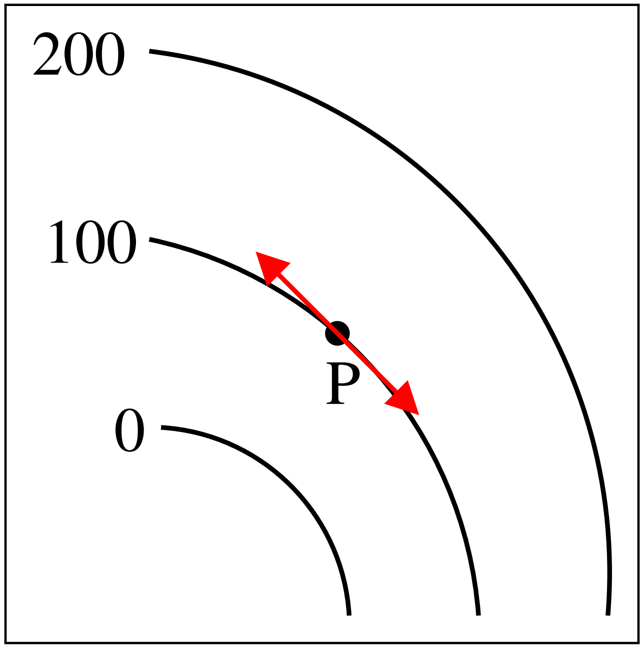


Contour map for part of a mountain with altitude $z = f(x, y)$

At point P, which direction $\vec{u}$ is best for each scenario?

- The **Power Hiker** wants the steepest uphill path.
- The **Power Skier** wants the steepest downhill path.
- The **Lazy Hiker** wants to avoid any elevation change.

The Lazy Hiker

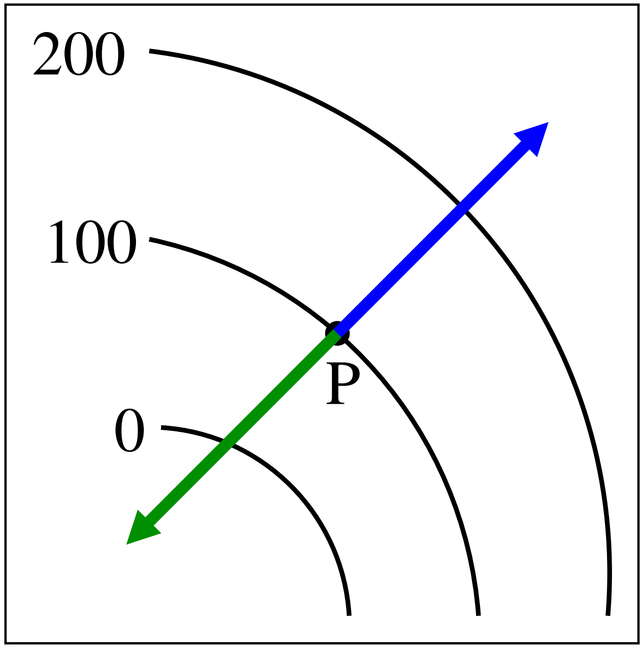


- To avoid elevation change, the lazy hiker walks along a level curve.
- At point P, the direction $\vec{u}$ is tangent to the level curve, giving the two options shown above.
- No elevation change along this path, so

$$D_{\vec{u}} f = 0 \qquad \text{so } \vec{u} \cdot \nabla f = 0 \qquad \text{so } \vec{u} \perp \nabla f$$

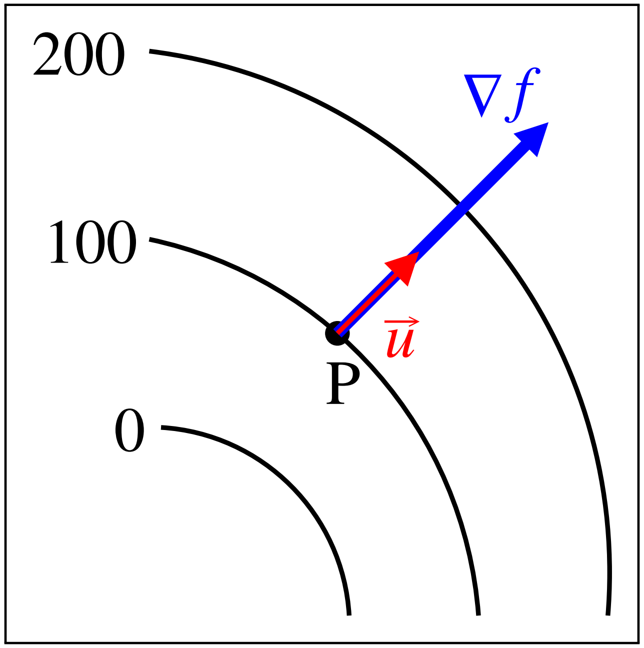
- So at any point $P = (a, b)$, the gradient $\nabla f(a, b)$ is perpendicular to the level curve.

Direction of gradient vector



$\nabla f(a,b)$ is perpendicular to the contour through $P=(a,b)$.
But which of these choices is it?

Power Hiker



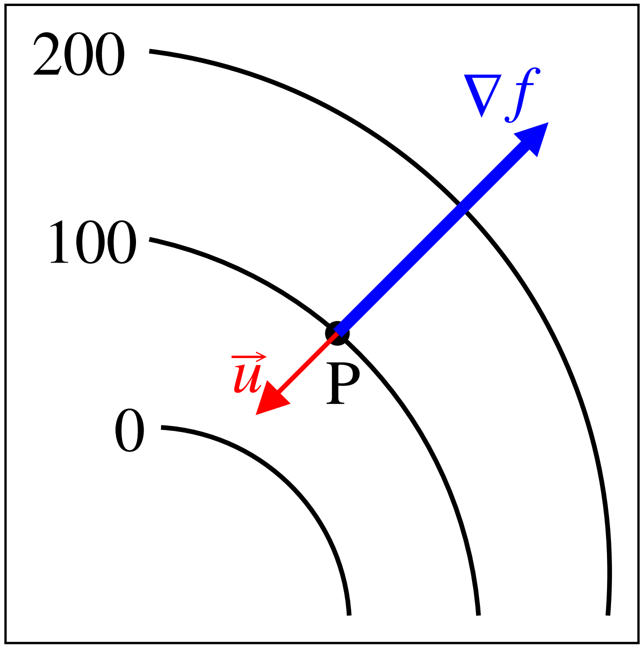
$$D_{\vec{u}}\, f = \vec{u} \cdot \nabla f = \|\vec{u}\|\ \|\nabla f\| \cos(\theta) = \|\nabla f\| \cos(\theta)$$

- As $\vec{u}$ varies, the maximum value of $D_{\vec{u}} f$ is $+\|\nabla f\|$.
- The maximum is when $\cos(\theta) = 1$, so $\theta = 0^\circ = 0$ radians.
- Thus, $\vec{u}$ is a unit vector in the same direction as ∇f, perpendicular to the level curve:

$$\vec{u} = \nabla f / \|\nabla f\|$$

- This is the *direction of steepest ascent*, or *fastest increase*.

Power Skier



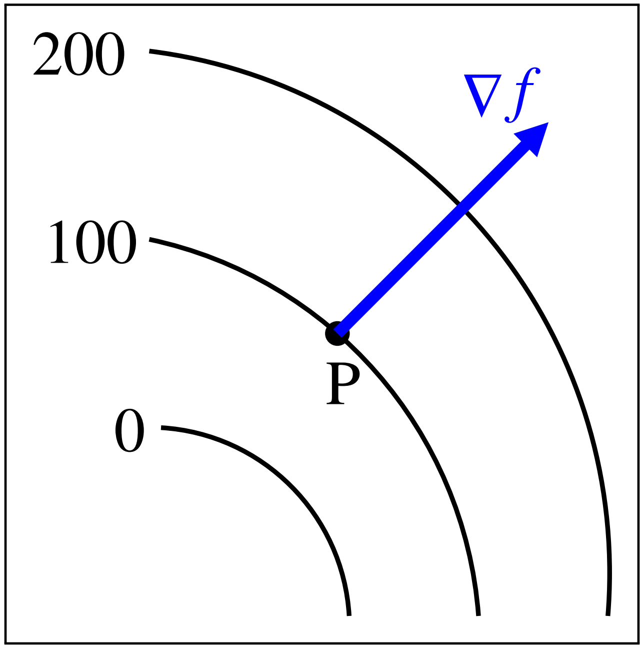
$$D_{\vec{u}} f = \vec{u} \cdot \nabla f = \|\vec{u}\| \; \|\nabla f\| \; \cos(\theta) = \|\nabla f\| \; \cos(\theta)$$

- As $\vec{u}$ varies, the minimum value of $D_{\vec{u}} f$ is $-\|\nabla f\|$.
- The minimum is when $\cos(\theta) = -1$, so $\theta = 180^\circ = \pi$ radians.
- Thus, $\vec{u}$ is a unit vector in the opposite direction of ∇f, still perpendicular to the level curve:

$$\vec{u} = -\nabla f / \|\nabla f\|$$

- This is the *direction of steepest decent*, or *fastest decrease*.

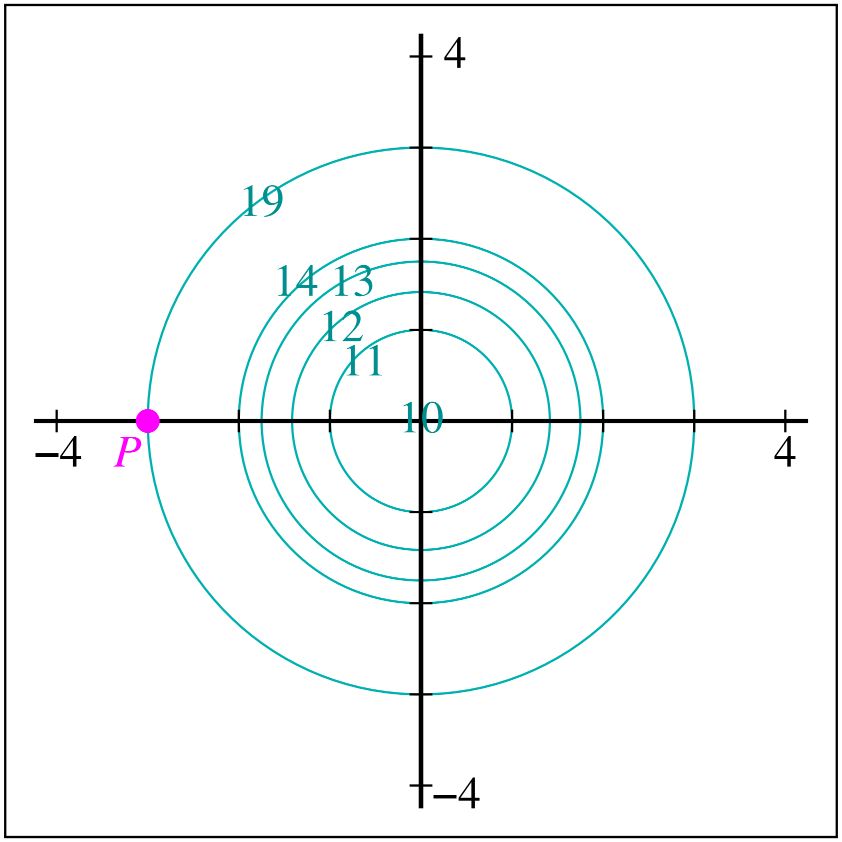
Direction of gradient vector



$\nabla f(a, b)$ is perpendicular to the contour through $P = (a, b)$.
It points to the side where f is increasing.

Example: $f(x, y) = x^2 + y^2 + 10$

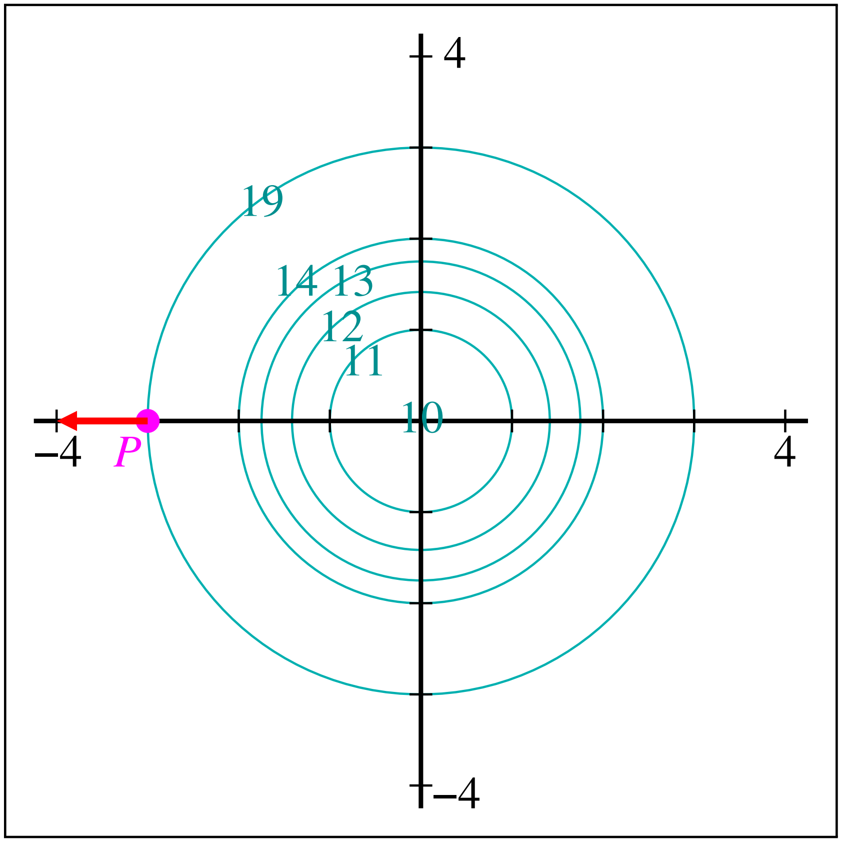
What is the direction of steepest ascent at point $P = (-3, 0)$?



- $\vec{u} = \nabla f / \|\nabla f\|$
- $\nabla f = \langle 2x, 2y \rangle$
- $\nabla f(-3, 0) = \langle -6, 0 \rangle$, with length $\|\nabla f(-3, 0)\| = 6$, so

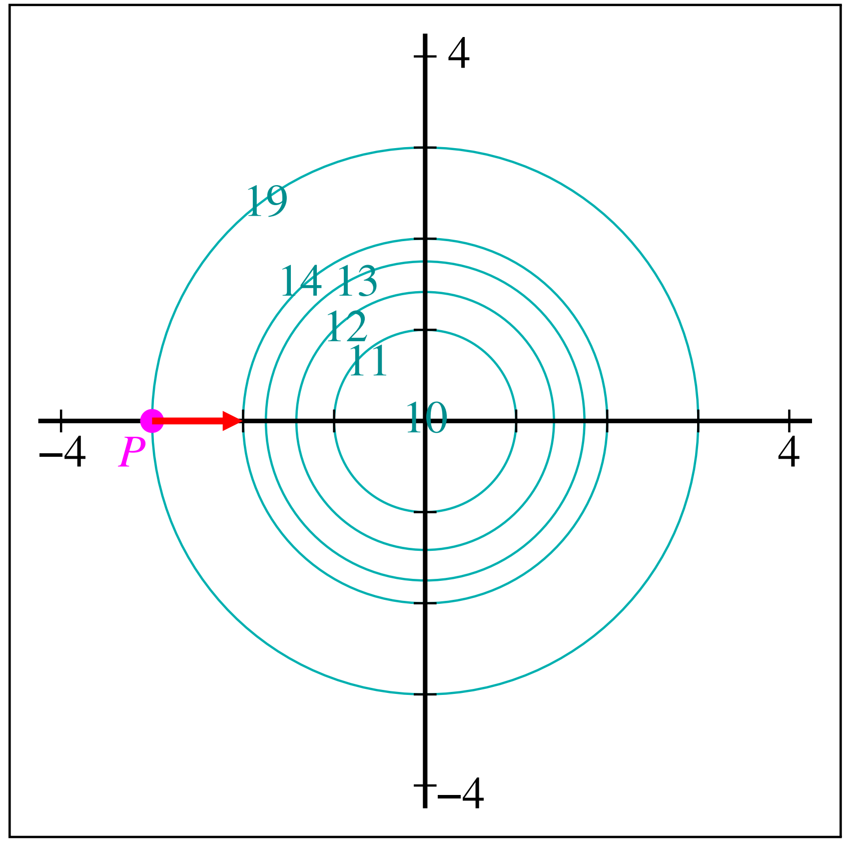
$$\vec{u} = \frac{\langle -6, 0 \rangle}{6} = \langle -1, 0 \rangle$$

Example: $f(x,y) = x^2 + y^2 + 10$



Steepest ascent

$\vec{u} = \langle -1, 0 \rangle$

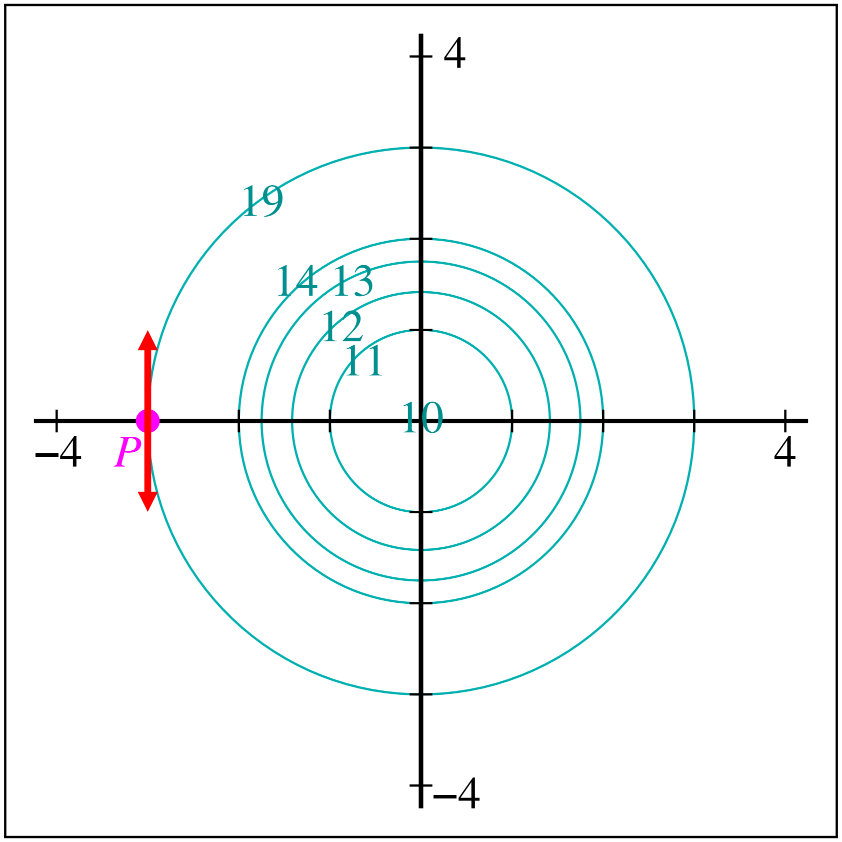


Steepest descent

$\vec{u} = -\langle -1, 0 \rangle = \langle 1, 0 \rangle$

Example: $f(x,y) = x^2 + y^2 + 10$

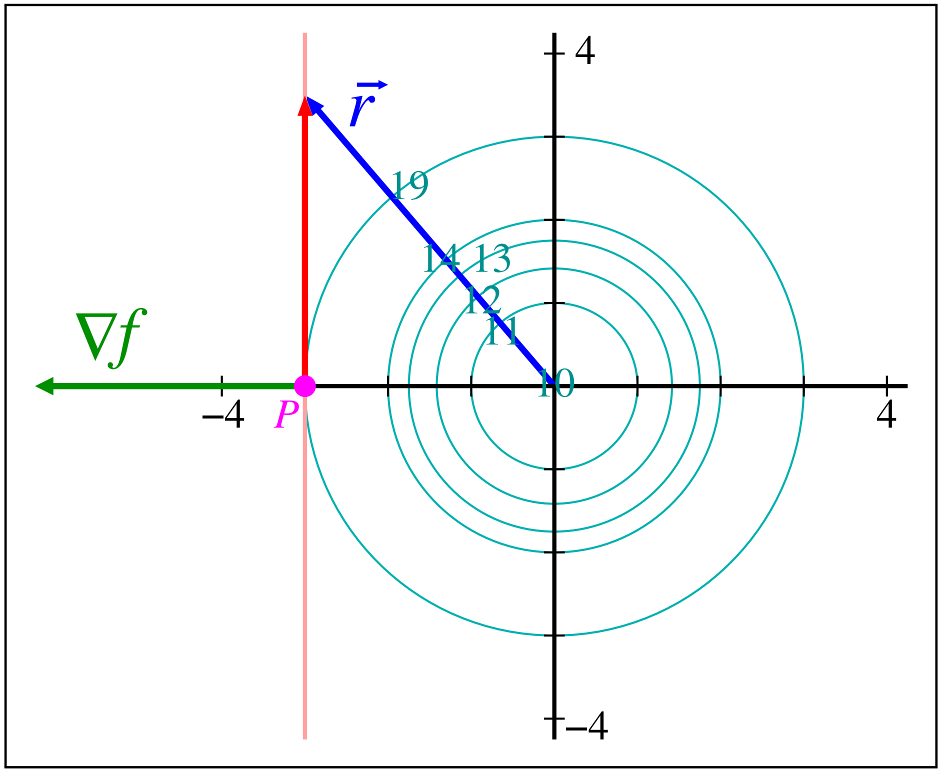
Direction of contour



- $\nabla f(-3,0) = \langle -6, 0 \rangle$ is perpendicular to the contour at point $(-3,0)$.
- In 2D, the directions $\perp \langle a,b \rangle$ are multiples of $\langle -b, a \rangle$ (or $\langle b, -a \rangle$).
- So $\langle -0, -6 \rangle$ is tangent to the contour.
- Unit vectors tangent to the contour are $\langle 0, \pm 1 \rangle$.

Example: $f(x,y) = x^2 + y^2 + 10$

Find the line tangent to the contour at $(-3,0)$



- Let $\vec{r}$ be a position vector along the line.
- The tangent line is $\perp$ $\nabla f(-3,0) = \langle -6, 0 \rangle$, so

$$\langle -6, 0 \rangle \cdot (\vec{r} - \langle -3, 0 \rangle) = 0$$

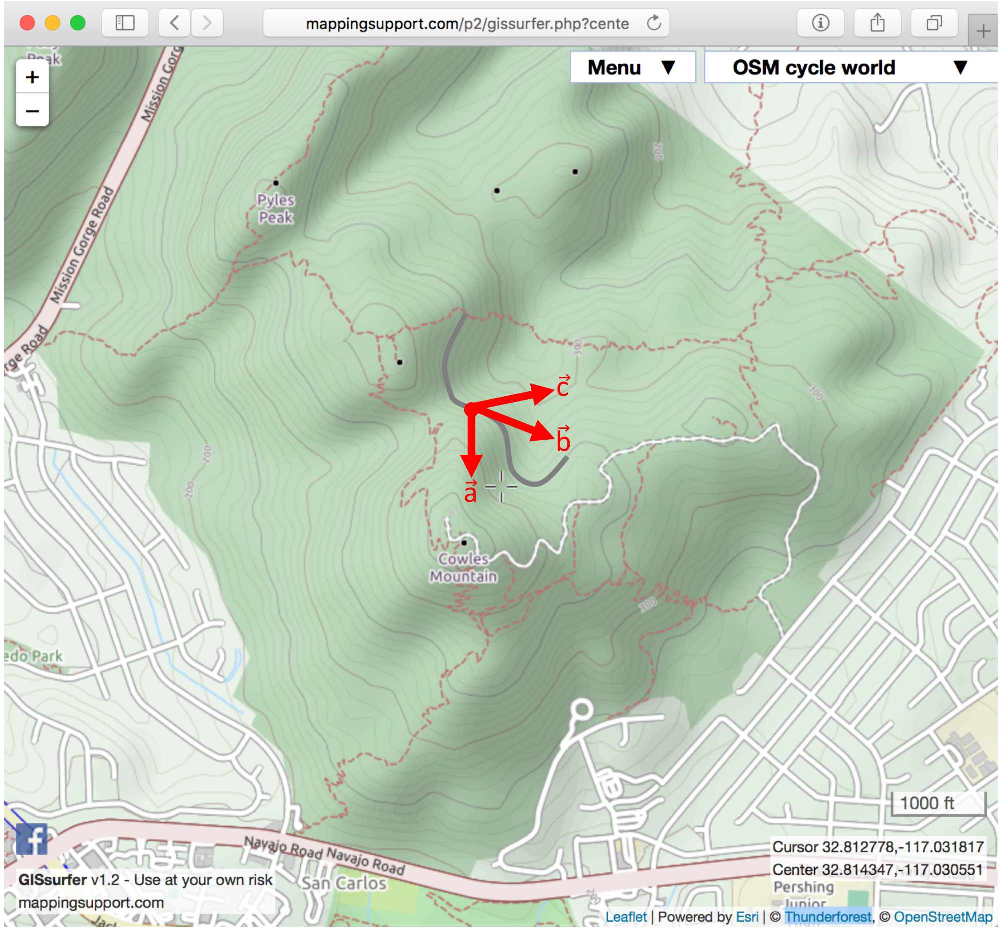
$$-6(x+3) + 0(y-0) = 0$$

$$-6(x+3) = 0$$

$$\boxed{\boldsymbol{x = -3}}$$

Topographic maps: Sign of $D_{\vec{u}} f$

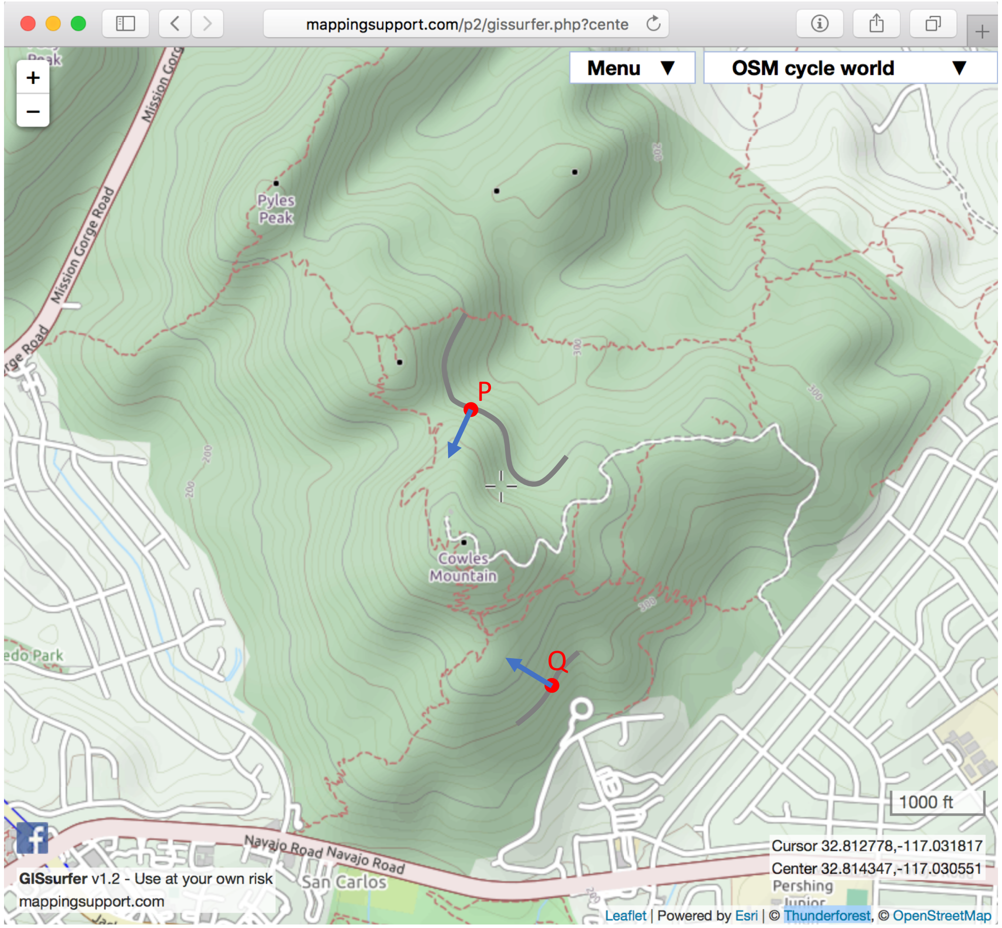
Screenshots from GISsurfer, mappingsupport.com, ©OpenStreetMap



- $\vec{a}$ points uphill, so $D_{\vec{a}} f > 0$ at the point shown.
- $\vec{b}$ is tangent to the contour, so $D_{\vec{b}} f = 0$.
- $\vec{c}$ points downhill, so $D_{\vec{c}} f < 0$.

Topographic maps: Signs of f_x and f_y

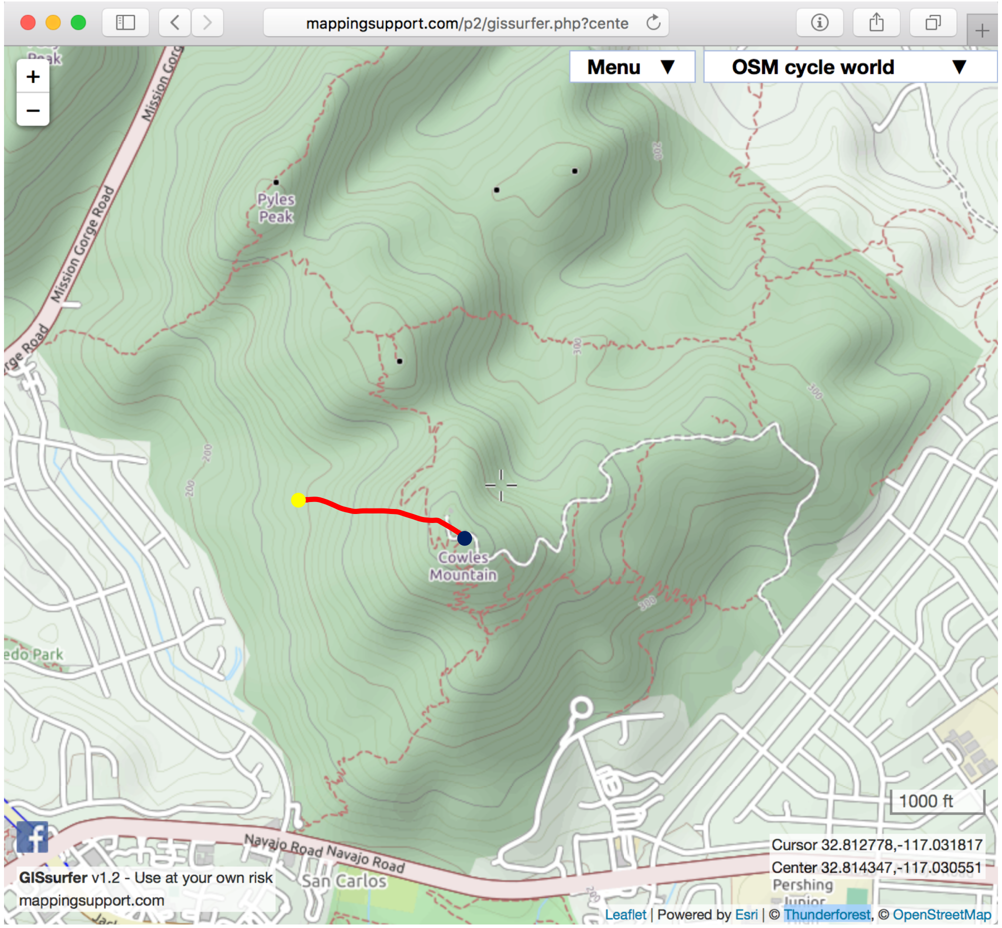
Screenshots from GISsurfer, mappingsupport.com, ©OpenStreetMap



- Gradients at P, Q are perpendicular to the contours on the uphill side.
- At $P = (a, b)$: $f_x(a, b) < 0$ and $f_y(a, b) < 0$.
- At $Q = (c, d)$: $f_x(c, d) < 0$ and $f_y(c, d) > 0$.

Topographic maps: Steepest ascent path

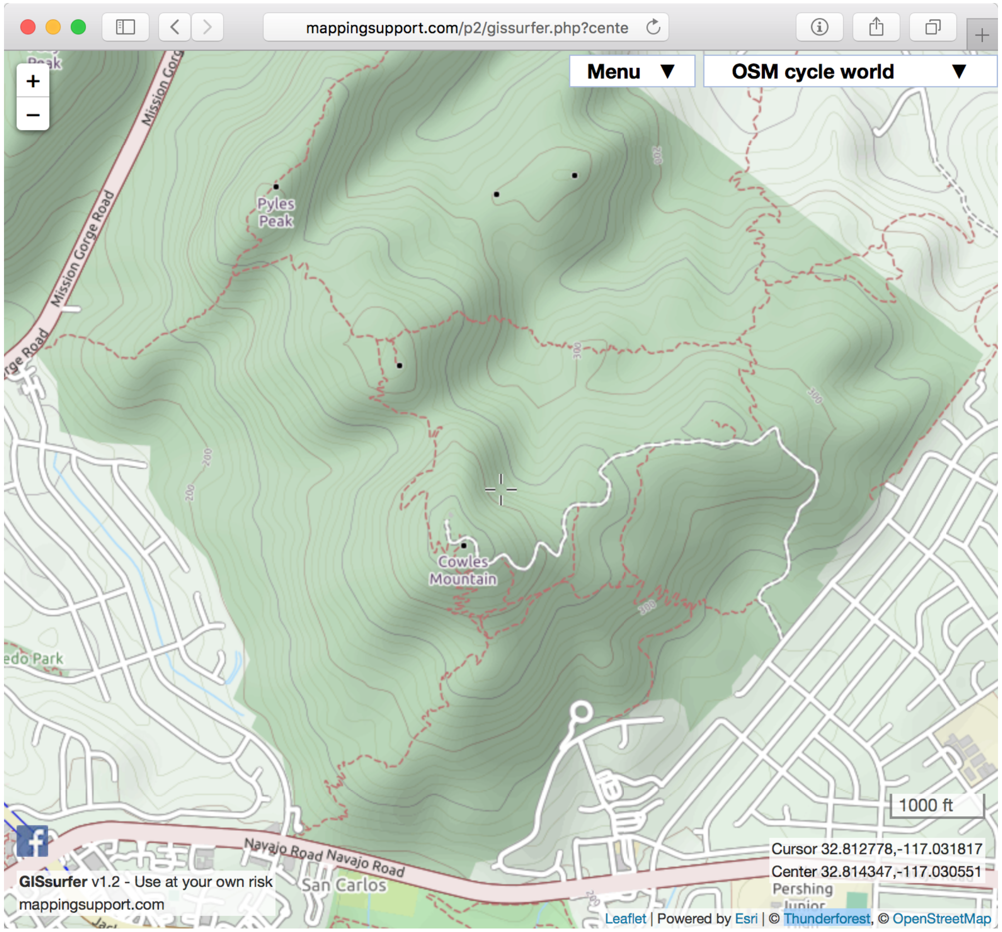
Screenshots from GISsurfer, mappingsupport.com, ©OpenStreetMap



- **Path of steepest ascent:** Draw a path starting at a point (yellow), continually adjusting direction to stay perpendicular to the contour in the uphill (increasing) direction.
- **Path of steepest descent:** Similar but going downhill.

Topographic maps: Maxima / Minima

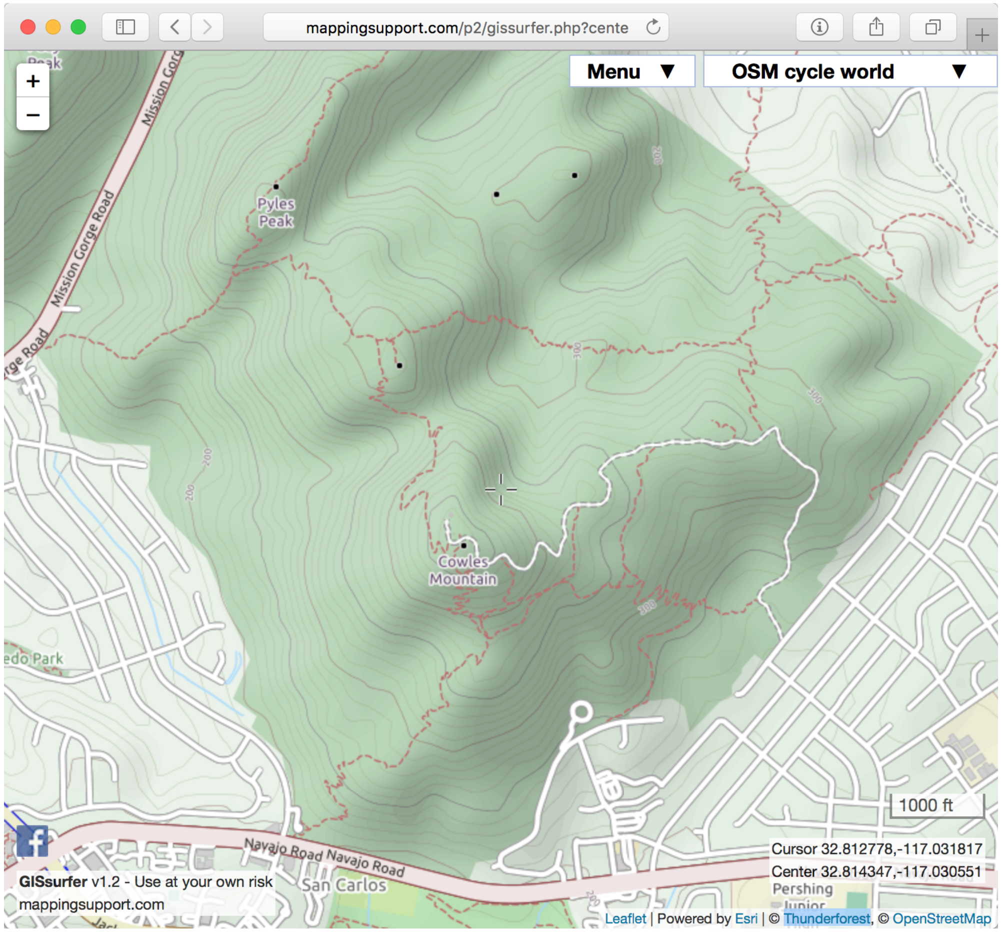
Screenshots from GISsurfer, mappingsupport.com, ©OpenStreetMap



- Contour map has closed curves encircling the mountain peaks (where the function is maximum).
- The same would happen with minimums.

Topographic maps: Switchbacks

Screenshots from GISsurfer, mappingsupport.com, ©OpenStreetMap



- It's steepest where the contours are closest together.
- The official hiking trails have switchbacks in the steepest regions.

Level surface of $f(x, y, z)$

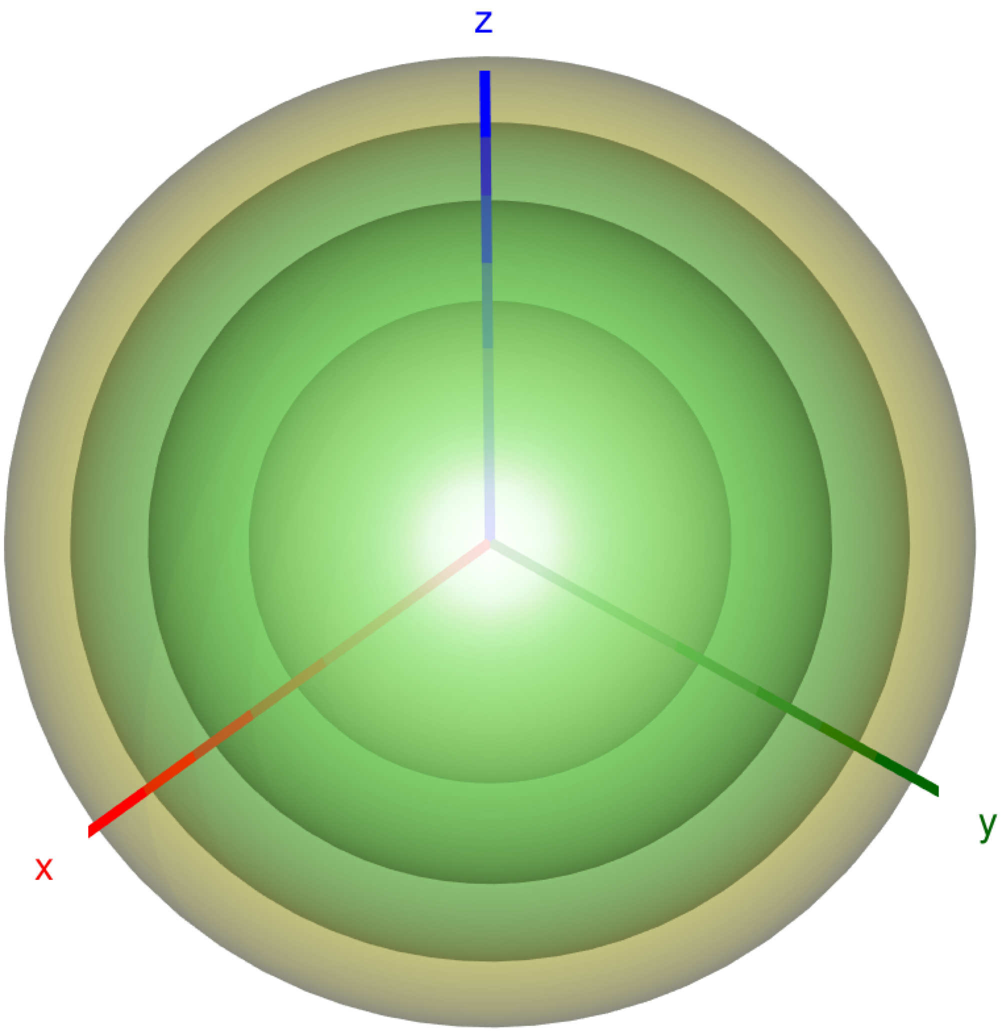
- For $z = f(x, y)$, contour maps have level curves $f(x, y) = k$.
  $\nabla f(a, b)$ is perpendicular to the level curve through (a, b).
- For $u = f(x, y, z)$, we get a *level surface* $f(x, y, z) = k$ instead of a level curve.
  $\nabla f(a, b, c)$ is perpendicular to the level surface through (a, b, c).

Example

- For $f(x, y, z) = x^2 + y^2 + z^2$, the level surface $f(x, y, z) = k$ is a sphere centered at $(0, 0, 0)$ of radius $\sqrt{k}$, provided $k \geqslant 0$.
- $\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$ is perpendicular to the sphere at (x, y, z).

Level surfaces of $f(x, y, z) = x^2 + y^2 + z^2$

Surfaces $f(x, y, z) = k$ shown for $k = 1, 2, 3, 4$ from inside to out



Level surface of $f(x, y, z)$

- Consider the surface

$$x^2 = 2x(y - z) + 9$$

- What is the point $(x, y, z) = (1, 2, \_\_)$?
Plug $x = 1$, $y = 2$ into the above equation, and solve for z:

$$\begin{aligned} 1^2 &= 2(1)(2 - z) + 9 \\ 1 &= 4 - 2z + 9 = 13 - 2z \\ 2z &= 13 - 1 = 12 \\ z &= 6 \end{aligned}$$

Level surface of $f(x, y, z)$

- Find the tangent plane to surface $x^2 = 2x(y-z)+9$ at $(x,y,z) = (1,2,6)$.
- Rearrange equation into $f(x, y, z) =$ constant:

$$x^2 - 2x(y-z) = 9 \qquad \text{so use } f(x,y,z) = x^2 - 2x(y-z).$$

- Normal vector:

$$\nabla f = \langle 2x - 2(y-z), -2x, 2x \rangle$$
$$\nabla f(1,2,6) = \langle 2(1) - 2(2-6), -2, 2 \rangle = \langle 10, -2, 2 \rangle$$

- Tangent plane $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$:

$$\boxed{\langle \mathbf{10}, \mathbf{-2}, \mathbf{2} \rangle \cdot (\vec{\boldsymbol{r}} - \langle \mathbf{1}, \mathbf{2}, \mathbf{6} \rangle) = \mathbf{0}}$$

$$\boxed{\mathbf{10}(\boldsymbol{x} - \mathbf{1}) - \mathbf{2}(\boldsymbol{y} - \mathbf{2}) + \mathbf{2}(\boldsymbol{z} - \mathbf{6}) = \mathbf{0}} \qquad \boxed{\mathbf{10}\boldsymbol{x} - \mathbf{2}\boldsymbol{y} + \mathbf{2}\boldsymbol{z} = \mathbf{18}}$$

Comparing tangent plane formulas from 2.3 vs. 2.6

2.3. Tangent plane to $z = f(x,y)$ at (x_0, y_0, z_0)

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

2.6. Tangent plane to $g(x,y,z) = k$ at (x_0, y_0, z_0)

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$, where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{n} = \nabla g(x_0, y_0, z_0)$.

This can be used even if you can't explicitly solve for z in terms of x, y.

Connection

$$z = f(x,y) \quad \text{is equivalent to} \quad \underbrace{z - f(x,y)}_{\text{call this } g(x,y,z)} = 0$$

- $\nabla g(x,y,z) = \left\langle -f_x, -f_y, 1 \right\rangle$.
- $\vec{n} = \nabla g(x_0, y_0, z_0) = \left\langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1 \right\rangle$
- The second formula $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ expands as
  $$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + 1(z - z_0) = 0,$$
  which is equivalent to the first formula.