Math 126 C
Worksheet 5

1. A **harmonic function** $u(x, y)$ is a function with continuous second partials which satisfies Laplace's equation,

$$u_{xx} + u_{yy} \equiv 0.$$

(a) Is $f(x, y) = x^2 - 2y^2$ harmonic?

Solution: $\boxed{\text{No}}$; $f_{xx} = 2$, $f_{yy} = -4$, and $2 - 4 = -2 \neq 0$.

(b) Let $g(x, y) = \ln(\sqrt{x^2 + y^2})$. Find the domain of g.

Solution: $\boxed{\text{The punctured plane}}$; we need $x^2 + y^2 > 0$ for the $\sqrt{x^2 + y^2}$ to make sense, so $(0, 0)$ isn't in the domain. The logarithm does not change this.

(c) Is $g(x, y)$ harmonic?

Solution: $\boxed{\text{Yes}}$; we compute

$$\begin{aligned} g_x &= \frac{1}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} \sqrt{x^2 + y^2} = \frac{1}{\sqrt{x^2 + y^2}} \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{x^2 + y^2} \\ g_{xx} &= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \end{aligned}$$

By symmetry, $g_{yy} = \frac{x^2 - y^2}{(x^2+y^2)^2}$, so $g_{xx} + g_{yy} = 0$.

(d) Find all local extrema for g.

Solution: Critical points occur when $g_x = \frac{x}{x^2+y^2} = 0$ and $g_y = \frac{y}{x^2+y^2} = 0$, i.e. when $x = y = 0$, which is not in the domain. Since each local extrema occurs at a critical point, there are $\boxed{\text{none}}$.

2. (a) Suppose u is a harmonic function with $u_{xx} \neq 0$ at each critical point. Can u have a local maximum?

Solution: $\boxed{\text{No}}$. Compute D from the Second Derivatives Test; note that $u_{xx} + u_{yy} = 0$ implies $u_{yy} = -u_{xx}$.

$$D = u_{xx}u_{yy} - u_{xy}^2 = -u_{xx}^2 - u_{xy}^2 < 0.$$

Thus every critical point occurs at a saddle point, not a local maximum (or minimum.) (If $u_{xx} = 0$, we might have $D = 0$ if u_{xy} were also 0, and the test would give no information.)

(b) A version of the *maximum principle* for harmonic functions states that a harmonic function achieves its absolute maximum on the boundary. Assume it for now.

Let $h(x, y) = e^x(\sin y+\cos y)$. Find the absolute maximum of this harmonic function on the square $\{(x, y) \mid |x| \leq 1, |y| \leq 1\}$, without computing a partial derivative of h. (Assume h is harmonic.)

Solution: By the maximum principle, we can check just the boundary of the square. This is composed of the four lines $(x, 1)$, $(x, -1)$, $(1, y)$, and $(-1, y)$ where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. On these lines, we have

$$\begin{aligned} h_1(x) &= h(x, 1) = e^x(\sin 1 + \cos 1) \\ h_2(x) &= h(x, -1) = e^x(\sin(-1) + \cos(-1)) \\ h_3(y) &= h(1, y) = e(\sin y + \cos y) \\ h_4(y) &= h(-1, y) = 1/e(\sin y + \cos y). \end{aligned}$$

One sees $h_1', h_2' \neq 0$ since $e^x \neq 0$, so the first two have no critical points. However, $h_3' = \cos y - \sin y$, so h_3 has a critical point at $y = \pi/4$. Likewise h_4 has a critical point at $y = \pi/4$. We have $h_3(\pi/4) = e\sqrt{2}$ and $h_4(\pi/4) = \sqrt{2}/e$. The boundaries of those curves are at $x = \pm 1, y = \pm 1$, and you can check numerically that $h(\pm 1, \pm 1)$ is always less than $\boxed{e\sqrt{2}}$. This occurs at $\boxed{(1, \pi/4)}$.

(c) Does your solution to (a) prove the maximum principle? Why or why not?

Solution: $\boxed{\text{No}}$. As noted in the solution to (a), it might happen that all the first partials of u vanish at a critical point, in which case $D = 0$ and the Second Derivatives Test gives no information. Indeed, $u(x, y) = x^4 - 6x^2y^2 + y^4$ is such a function. $(0, 0)$ is a critical point, and every second partial is 0 here.

However, when (a) applies, we find that an absolute maximum must occur on the boundary, since otherwise it would give a critical point, but by (a) all such points are saddle points, not maximums. So, (a) “almost” gives the maximum principle.

3. (a) Compute the tangent plane to h at the point from (2b).

Solution: Using the tangent plane formula at $(1, \pi/4)$,

$$z - h(1, \pi/4) = h_x(1, \pi/4)(x-1) + h_y(1, \pi/4)(y - \pi/4).$$

We have

$$h_x = e^x(\sin y + \cos y)$$
$$h_y = e^x(\cos y - \sin y),$$

so plugging things in to the above gives

$$z - e\sqrt{2} = e\sqrt{2}(x-1) + 0(x - \pi/4),$$

which simplifies to just $\boxed{z = xe\sqrt{2}}$.

(b) In which direction should I travel to increase h the fastest, starting at the point from (2b)? Can this direction be towards the origin? Why or why not?

Solution: Going in the y-direction gives a slope of 0 since $h_y(1, \pi/4) = 0$. The highest rate of increase must be in the $\pm x$-direction. Since $h_x(1, \pi/4) > 0$, it's the $\boxed{+x\text{-direction}}$.
Alternatively, consider a direction specified by $(\Delta x, \Delta y)$ of unit length, i.e. $(\Delta x)^2 + (\Delta y)^2 = 1$. We have the differential centered at $(1, \pi/4)$ as $\Delta h = (\Delta x)e\sqrt{2}$. For what value of Δx is this as large as possible, subject to the above constraint? $\Delta x = 1$, indicating we can increase h the most by moving in the $+x$-direction, as suggested above.

4. (a) Find a (non-linear) polynomial $p(x, y)$ with the same tangent plane as h at the point from (2b).

Solution: We can start with the tangent plane $z = xe\sqrt{2}$ and add something whose value and first partials all vanish at $(1, \pi/4)$: $(x-1)^2 + (y - \pi/4)^2$ works. That is, $\boxed{p(x, y) = xe\sqrt{2} + (x-1)^2 + (y - \pi/4)^2}$ works.

(b) Repeat (a), but make the tangent planes agree at both the point from (2b) and at the origin.

Solution: We need to compute the tangent plane at the origin. Using (3a)'s techniques, this is $z - 1 = x + y$. For h and p to have the same tangent planes at $(0,0)$ and $(1,\pi/4)$, we need $p(0,0)=1$, $p_x(0,0)=1$, $p_y(0,0)=1$; $p(1,\pi/4)=e\sqrt{2}$, $p_x(1,\pi/4)=e\sqrt{2}$, $p_y(1,\pi/4)=0$.

There are several ways to proceed. One is as follows: we had quite a bit of freedom in our solution to (a). We could actually have added $a(x-1)^2+b(y-\pi/4)^2+c(x-1)^3$ for arbitrary constants a,b,c without changing the tangent plane at $(1,\pi/4)$. We have three additional constraints to get the tangent plane at $(0,0)$ correct, so we should be able to choose a,b,c to satisfy them all. Let $p(x,y)=xe\sqrt{2}+a(x-1)^2+b(y-\pi/4)^2+c(x-1)^2(y-\pi/4)^2$, so we have

$$\begin{aligned} p(0,0) &= a+\frac{b\pi^2}{16}+\frac{c\pi^2}{16}=1 \\ p_x(0,0) &= -2a-\frac{c\pi^2}{8}+e\sqrt{2}=1 \\ p_y(0,0) &= -\frac{b\pi}{2}-\frac{c\pi}{2}=1. \end{aligned}$$

One may solve this system and get

$$\begin{aligned} a &= 1+\frac{\pi}{8} \\ b &= \frac{24-8e\sqrt{2}}{\pi^2} \\ c &= \frac{-24+8e\sqrt{2}}{\pi^2}-\frac{2}{\pi}. \end{aligned}$$

In all, the following has the specified tangent planes:

$$\begin{aligned} p(x,y) = xe\sqrt{2} &+ \left(1+\frac{\pi}{8}\right)(x-1)^2+\frac{24-8e\sqrt{2}}{\pi^2}(y-\pi/4)^2 \\ &+ \left(\frac{-24+8e\sqrt{2}}{\pi^2}-\frac{2}{\pi}\right)(x-1)^2(y-\pi/4)^2. \end{aligned}$$

(c) Use differentials to approximate the difference between your polynomial and h at $(0.1,0.1)$. Does your polynomial approximate h well near the origin?

Solution: Differentials use the tangent plane approximation, and the closest convenient point to $(0.1,0.1)$, $(0,0)$, was chosen so that p and h have the same tangent planes there. Thus differentials will estimate the difference between the two as $\boxed{0}$—a more subtle approach is needed. A "second-order" approximation would work, though it's beyond the scope of this course, so we'll just note that $p(0.1,0.1)\approx 1.29705$ while $h(0.1,0.1)\approx 1.20998$, so it seems the approximation is relatively good.