Math 126 C
Worksheet 5

1. A **harmonic function** $u(x, y)$ is a function with continuous second partials which satisfies Laplace's equation,
$$u_{xx} + u_{yy} \equiv 0.$$
   (a) Is $f(x, y) = x^2 - 2y^2$ harmonic?

   (b) Let $g(x, y) = \ln(\sqrt{x^2 + y^2})$. Find the domain of g.

   (c) Is $g(x, y)$ harmonic?

   (d) Find all local extrema for g.

2. (a) Suppose u is a harmonic function with $u_{xx} \neq 0$ at each critical point. Can u have a local maximum?

(b) A version of the *maximum principle* for harmonic functions states that a harmonic function achieves its absolute maximum on the boundary. Assume it for now.

Let $h(x, y) = e^x(\sin y + \cos y)$. Find the absolute maximum of this harmonic function on the square $\{(x, y) \mid |x| \leq 1, |y| \leq 1\}$, without computing a partial derivative of h. (Assume h is harmonic.)

(c) Does your solution to (a) prove the maximum principle? Why or why not?

3. (a) Compute the tangent plane to h at the point from (2b).

   (b) In which direction should I travel to increase h the fastest, starting at the point from (2b)? Can this direction be towards the origin? Why or why not?

4. (a) Find a (non-linear) polynomial $p(x, y)$ with the same tangent plane as h at the point from (2b).

   (b) Repeat (a), but make the tangent planes agree at both the point from (2b) and at the origin.

   (c) Use differentials to approximate the difference between your polynomial and h at $(0.1, 0.1)$. Does your polynomial approximate h well near the origin?