Math 126 C Worksheet 5 Solutions

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Note: Send corrections, if any, to `jps314@math.washington.edu`.

(1) For simplicity say the circle is centered at the origin and lies in the plane. A parametrization coming from polar coordinates is given by $\mathbf{p}(t) = (r\cos(t), r\sin(t))$. For later convenience, compute

$$\mathbf{p}'(t) = \langle -r\sin(t), r\cos(t)\rangle$$
$$\mathbf{p}''(t) = \langle -r\cos(t), -r\sin(t)\rangle$$

The speed is indeed constant: $|\mathbf{p}'(t)| = \sqrt{r^2\sin^2(t) + r^2\cos^2(t)} = \sqrt{r^2} = r$. The direction of travel is $\mathbf{p}'(t)$, which is perpendicular to the acceleration since $\mathbf{p}'(t)\cdot\mathbf{p}''(t) = r^2\sin(t)\cos(t) - r^2\cos(t)\sin(t) = 0$.

At $t = 0$, we have $\mathbf{p}(0) = (r, 0)$, $\mathbf{p}'(0) = \langle 0, r\rangle$, and $\mathbf{p}''(0) = \langle -r, 0\rangle$. If you draw these three vectors out, you'll find $\mathbf{p}''(0)$ is pointing toward the center of the circle, so it indeed points into the circle. The same is true for every other value of t.

(2) On a straight line segment of a track, there is no acceleration (assuming the train is traveling at constant speed, as in (a)). On a circular segment, there is acceleration perpendicular to the direction of motion. At a juncture of straight and circular track, to stay on the track the train's acceleration must suddenly go from zero to some large amount. This imparts a huge impulse to the track, likely breaking it, causing the train to continue along its original, linear path, so it leaves the track. (A ridiculously well-made track would theoretically be strong enough to force the train to stay on it.)

(3) Let $\mathbf{L}_1(x)$ be the curve on the line $y = -x$ for $x \leq -1$ and let $\mathbf{L}_2(x)$ be the curve on the line $y = x$ for $x \geq 1$. We want a new curve $\mathbf{S}(x)$ for $-1 \leq x \leq 1$ with nice properties. Explicitly,

$$\mathbf{L}_1(x) = (x, -x)$$
$$\mathbf{L}_2(x) = (x, x)$$
$$\mathbf{S}(x) = (x, y(x))$$

where $y(x)$ is a function we will determine.

For the curves to match up, we need $(-1, 1) = \mathbf{L}_1(-1) = \mathbf{S}(-1) = (-1, y(-1))$, i.e. $y(-1) = 1$, and similarly $y(1) = 1$. For the curves to "connect smoothly", we need their tangent vectors to agree at the points where they connect. We have

$$\mathbf{L}_1'(x) = \langle 1, -1\rangle$$
$$\mathbf{L}_2'(x) = \langle 1, 1\rangle$$
$$\mathbf{S}'(x) = \langle 1, y'(x)\rangle$$

Thus we need $\langle 1, -1\rangle = \mathbf{L}_1'(-1) = \mathbf{S}'(-1) = \langle 1, y'(-1)\rangle$, so $y'(-1) = -1$, and similarly $y'(1) = 1$. Further, we want to make the transition with "no sudden changes in acceleration", so we want $(0, 0) = \mathbf{L}_1''(-1) = \mathbf{S}''(-1) = (0, y''(-1))$, i.e. $y''(-1) = 0$, and similarly $y''(1) = 0$.

(4) (a) Geometrically, an even polynomial is symmetric about the y-axis, and since the problem is symmetric about the y-axis, you would naturally expect even polynomials to simplify the search.

(b) An even polynomial of degree 4 is generally of the form

$$y(x) = ax^4 + bx^2 + c.$$

Our conditions on y are $y(\pm 1) = 1$, $y'(\pm 1) = \pm 1$, and $y''(\pm 1) = 0$. We see $y'(x) = 4ax^3 + 2bx$ and $y''(x) = 12ax^2 + 2b$. Plugging in $x = \pm 1$ and using these conditions gives

$$\begin{aligned} y(\pm 1) &= a + b + c = 1 \\ y(\pm 1) &= \pm(4a + 2b) = \pm 1 \\ y(\pm 1) &= 12a + 2b = 0 \end{aligned}$$

We may cancel the $\pm$'s, which gives a system of three equations in three unknowns,

$$\begin{aligned} a + b + c &= 1 \\ 4a + 2b &= 1 \\ 12a + 2b &= 0. \end{aligned}$$

Using one of the standard techniques, one finds the unique solution as $a = -(1/8), b = 3/4, c = 3/8$, so the polynomial is

$$y(x) = -(1/8)x^4 + (3/4)x^2 + (3/8).$$

A plot of this function shows that it is plausible.

(5) Recall the formula for the curvature of a graph:

$$\kappa(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}}.$$

Using the function above, we have $y'(x) = -(1/2)x^3 + (3/2)x$, so $y''(x) = -(3/2)x^2 + (3/2)$, so the curvature is

$$\kappa(x) = \frac{|-(3/2)x^2 + (3/2)|}{[1 + (-(1/2)x^3 + (3/2)x)^2]^{3/2}}.$$

This may simplify, though we only need to compute the limit as $x \to \pm 1$. Note that $\kappa(x) = \kappa(-x)$ from the above formula, so we just compute

$$\begin{aligned} \lim_{x \to 1} \kappa(x) &= \frac{|-(3/2) + (3/2)|}{[1 + (-(1/2) + (3/2))^2]^{3/2}} \\ &= 0. \end{aligned}$$

(The denominator is non-zero at $x = 1$, so we don't need to use L'Hopital's rule or similar techniques; we can just plug in $x = 1$.)