Math 126 C Challenge Problems/Solutions

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1. Compute the Taylor series about $b = 0$ for $f(x) = (1+x)^a$, where a is a fixed constant.

Compute derivatives:

$$\begin{aligned} f(x) &= (1+x)^a \\ f'(x) &= a(1+x)^{a-1} \\ f''(x) &= a(a-1)(1+x)^{a-2} \\ &\vdots \\ f^k(x) &= a(a-1)\cdots(a-k+1)(1+x)^{a-k}. \end{aligned}$$

Evaluated at 0, we have $f^k(0) = a(a-1)\cdots(a-k+1)$. For convenience, make a definition:

$$\binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!} = \frac{f^{(k)}(0)}{k!},$$

where a is a fixed constant and $k \geq 0$ is an integer. This is called a *generalized binomial coefficient*; the more elementary *binomial coefficient* $\binom{n}{k}$ assumes n is an integer but uses the same definition. In that case, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$; (2) below will give a physical interpretation of a particular binomial coefficient.

In any case, the Taylor series for $(1+x)^a$ based at $b = 0$ using the above calculation is

$$\sum_{k=0}^{\infty} \binom{a}{k} x^k.$$

One may show that this converges for $|x| < 1$ no matter which a is used, and it sometimes for other values of x, depending on the constant a. ■

2. Use (1) to compute the number of ways for a group of 1000 people to choose a committee consisting of 700 of those people. (Hint: Let $a = 1000$ in (1) and look at the 700th term.)

Using the hint, the series becomes

$$\sum_{k=0}^{\infty} \binom{1000}{k} x^k.$$

Note that $n(n-1)\cdots(n-k+1) = 0$ when one of the factors is 0, which occurs when n is an integer greater than k. That is, the series is all zeros past the $k = n$ term. This also implies

$$(1+x)^{1000} = \sum_{k=0}^{1000} \binom{1000}{k} x^k.$$

That is, we've found a way to "expand" the polynomial. The hint suggests we look at the $k = 700$ term, $\binom{1000}{700}x^{700}$. When expanding $(1+x)^{1000}$, what will contribute to this term? To expand $(1+x)(1+x)\cdots(1+x)$, we pick either 1 or x from each of the 1000 factors, multiply them up, and sum over all possible choices. (For instance, $(1+x)(1+x) = 1\cdot 1 + 1\cdot x + x\cdot 1 + x\cdot x$—the $1\cdot x$ term came from choosing 1 in the first factor and x in the second factor.) The only way to get x^{700} in a term of this sum is to choose precisely 700 of the x's.

Imagine each of the 1000 linear factors corresponds to a particular person in the group of 1000 people. Choosing a committee consisting of 700 of those people is then precisely the same as choosing 700 of the x's. The number of ways to perform this choice is then the coefficient on the x^{700} term in the expanded form of $(1+x)^{1000}$. We computed this using the Taylor series as $\binom{1000}{700}$.

In all, $\binom{1000}{700} = \frac{1000\cdot 999\cdots 301}{300!}$ is the number of ways to choose such a committee. (This is a number with 264 digits.) ∎