Math 126 C Challenge Problems/Solutions

Problems Posted 8/19/2013
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1. Use Taylor series to prove *Euler's formula*,

$$e^{ix} = \cos x + i \sin x.$$

Use this to determine the five complex 5th roots of 1. (That is, complex numbers z such that $z^5 = 1$.)

We see

$$\begin{aligned}
e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + i \sin x.
\end{aligned}$$

Some of these manipulations require more justification. The most serious leap is the first equality; are we allowed to plug complex numbers in to Taylor series? What does e^{ix} even mean? A common solution is to sidestep this difficulty by *defining* e^{ix} using the series above. The only trouble with this approach is that standard exponential identities are no longer immediately obvious: is $e^{ix+iy} = e^{ix}e^{iy}$ using the series above? Yes, though the proof is beyond the scope of this writeup. If you're interested in such rigorous justification, take a course in real analysis.

As for the fifth roots, we see $(e^{ix})^5 = e^{5ix} = \cos(5x) + i\sin(5x) = 1 + 0i$ forces $\cos(5x) = 1$, $\sin(5x) = 0$. This in turn forces $x = 0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5$ (or multiples of 2π more or less than one of these). The roots are then

$$\begin{aligned}
e^{0i} &= \cos(0) + i\sin(0) = 1 \\
e^{2\pi i/5} &= \cos(2\pi/5) + i\sin(2\pi/5) = \frac{\sqrt{5}-1}{4} + i\sqrt{\frac{5+\sqrt{5}}{8}} \\
e^{4\pi i/5} &= \cos(4\pi/5) + i\sin(4\pi/5) = -\frac{\sqrt{5}+1}{4} + i\sqrt{\frac{5-\sqrt{5}}{8}} \\
e^{6\pi i/5} &= \cos(6\pi/5) + i\sin(6\pi/5) = -\frac{\sqrt{5}+1}{4} - i\sqrt{\frac{5-\sqrt{5}}{8}} \\
e^{8\pi i/5} &= \cos(8\pi/5) + i\sin(8\pi/5) = \frac{\sqrt{5}-1}{4} - i\sqrt{\frac{5+\sqrt{5}}{8}}
\end{aligned}$$

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2. Use Euler's formula to prove *Euler's identity*,

$$e^{i\pi} + 1 = 0.$$

Also use Euler's formula to show

$$8\cos(20^\circ)^3 - 6\cos(20^\circ) - 1 = 0.$$

(This expression comes up in proving the impossibility of trisecting a 60° angle using only a compass and straightedge. One may show that such a trisection would force the polynomial $8x^3 - 6x - 1$ to have a rational root; it doesn't.)

Euler's formula gives

$$e^{i\pi} = \cos\pi + i\sin\pi = -1,$$

giving Euler's identity immediately. For the $\cos(20^\circ)$ part, first note that Euler's formula gives

$$\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \\ \Rightarrow \frac{e^{ix}+e^{-ix}}{2} &= \cos x. \end{aligned}$$

This gives us a way to reduce $\cos^3 x$:

$$\begin{aligned} \cos^3(x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 = \frac{e^{3ix} + 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} + e^{-3ix}}{8} \\ &= \frac{1}{4}\frac{e^{3ix}+e^{-3ix}}{2} + \frac{3}{4}\frac{e^{ix}+e^{-ix}}{2} = \frac{\cos(3x) + 3\cos(x)}{4}, \end{aligned}$$

i.e.

$$4\cos^3(x) = \cos(3x) + 3\cos(x).$$

The given expression is then

$$\begin{aligned} 8\cos(20^\circ)^3 - 6\cos(20^\circ) - 1 &= 2(\cos(60^\circ) + 3\cos(20^\circ) - 6\cos(20^\circ) - 1 \\ &= 2\cos(60^\circ) - 1 \\ &= 2(1/2) - 1 = 0. \end{aligned}$$

(For those counting, that's 5 variations on "gives" in one writeup.) ∎