Math 126 C Challenge Problems/Solutions

Problems Posted 8/16/2013
Solutions Posted 8/19/2013

1. Compute $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{1024}$ exactly. Deduce

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

for $-1 < x < 1$.

You can verify by hand that $1+\frac{1}{2}+\ldots+\frac{1}{1024} = \frac{2047}{1024}$. The numerator is very close to $2048 = 2\times 1024 = 2\cdot 2^{10}$. In general, one might then expect

$$1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = \sum_{k=0}^{n} \frac{1}{2^k} = \frac{2\cdot 2^n - 1}{2^n} = 2 - \frac{1}{2^n}.$$

One would be right. How do you prove it? There are a few options.

- Draw the situation out geometrically. You start at 0, add 1 to get to 1, add $\frac{1}{2}$ to get to $\frac{3}{2}$, etc. After each addition, you can see from the picture that you're just the amount you most recently added away from 2, which is precisely the formula above.
- An algebraic version of the previous argument then suggests itself: we know adding $\frac{1}{2^n}$ should give us 2, so do so and try to simplify:

$$\begin{aligned}
1 + \frac{1}{2} + \ldots + \left(\frac{1}{2^n} + \frac{1}{2^n}\right) &= 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} + \left(\frac{1}{2^{n-1}}\right) \\
&= 1 + \frac{1}{2} + \ldots + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}}\right) \\
&= 1 + \frac{1}{2} + \ldots + \frac{2}{2^{n-2}} + \left(\frac{2}{2^{n-1}}\right) \\
&= \ldots \\
&= 1 + \left(\frac{1}{2} + \frac{1}{2}\right) \\
&= 2.
\end{aligned}$$

- Another clever method is *mathematical induction*. In general, say you have an infinite sequence of statements to prove. You can do so by proving two potentially easier things: (1) the first statement in the sequence is true, and (2) *if* one statement is true, *then* the next statement is true. This builds up a "ladder" of implications, where, for instance, the first statement is true by (1), so the second statement is true by applying (2) starting with the first statement, hence the third statement is true by applying (2) starting with the second statement, etc. Here's a proof using induction of the above fact:

  (1) For $n = 1$, we have $1 + \frac{1}{2} = \frac{3}{2} = 2 - \frac{1}{2}$ by simple arithmetic. An even simpler starting case here happens to be $n = 0$, where the statement is just $1 = 2 - 1$.

(2) Assume that $1 + \frac{1}{2} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ is true. This is the "inductive hypothesis". Then we have

$$\left(1 + \frac{1}{2} + \ldots + \frac{1}{2^n}\right) + \frac{1}{2^{n+1}} = \left(1 - \frac{1}{2^n}\right) + \frac{1}{2^{n+1}}$$
$$= 1 - \frac{1}{2^{n+1}}$$

which is precisely the desired statement in the next case.

Proofs by induction ultimately allow you to say, "and repeat the process at most finitely many times" in a proof, but it's often clearer to do it this way, you don't have to write ... everywhere, and induction can reduce your "cognitive load" while finding a proof.

Taking $n \to \infty$ in the above gives the expression

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Similar arguments give

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$$

for all $x \neq 1$, so that, if $|x| < 1$,

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x}$$
$$= \frac{1 - 0}{1 - x},$$

which is the claim. Indeed, this limit exists for $x \neq 1$ if and only if $\lim_{n\to\infty} x^{n+1}$ exists, which occurs if and only if $|x| < 1$. (The sum is clearly divergent for $x = 1$.)

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2. Compute the Taylor series for $f(x) = \ln(x)$ about b for any positive real number b. For what values of x does this converge?

Compute a few derivatives:

$$f^{(0)}(b) = \ln(b)$$
$$f'(b) = \frac{1}{b}$$
$$f''(b) = \frac{-1}{b^2}$$
$$f'''(b) = \frac{2}{b^3}$$
$$\ldots$$
$$f^{(n)}(b) = (-1)^{n+1} \frac{(n-1)!}{b^n} \quad \text{[for } n \geq 1\text{]}$$

The infinite Taylor series is then

$$\ln(b) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{b^n n} (x - b)^n$$

If $x = 0$, the sum becomes the harmonic series, which from last week's challenge problems diverges. So, $x > 0$ is needed for convergence, at least, and in particular the question of where the series converges is subtle.

We can use the geometric series cleverly to show that the above converges for $|x - b|/|b| < 1$, though this is formalized in the following standard test which we'll use instead:

Proposition 1 (Ratio Test (Special Case)) *Let a_n be a sequence of real numbers. If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists and is less than 1, then $\sum_n a_n$ converges.* □

Using this test on the above sum, the limit can easily be computed to be $|x - b|/|b|$, giving the claimed result above. Rearranging, the series converges (at least) for $|x - b| < |b|$, i.e. for x in a circle of radius b around b. Indeed, our reasoning can be made to work even if x is a complex number, in which case this circle interpretation is the correct one. In any case, using real numbers, we have convergence for $0 < x < 2b$.

For the remaining pieces, another part of the ratio test, using essentially the same reasoning, shows that the sum diverges for $x < 0$ and $2b < x$. We already settled $x = 0$ above. As for $x = 2b$, it turns out we get the *alternating harmonic series*, which converges by the *alternating series test*, which I won't include here. It happens to have value $\ln(2)$, so the entire approximation is $\ln(b) - \ln(2) = \ln(b/2) = \ln(x)$, which is actually correct. This is a manifestation of Abel's limit theorem from a previous challenge set.

To conclude, the series converges precisely for x in $(0, 2b]$. ■