Math 126 C Challenge Problems/Solutions

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1. Compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ exactly. (Hint: consider double integrals.)

We can integrate xe^{-x^2} by substitution, but not e^{-x^2}. Still, the exponent is part of $-x^2-y^2=-r^2$ in polar, and when switching to polar we pick up an extra factor of r. That is,

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\,dr\,d\theta \\ &= 2\pi \left.\frac{e^{-r^2}}{-2}\right|_{r=0}^{\infty} = \pi.\end{aligned}$$

But on the other hand, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2.\end{aligned}$$

Comparing these computations and taking a square root gives

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.}$$

This integral is called the Gaussian integral. ■

2. Give a physical interpretation to $\iint_R x^2\rho(x,y)\,dA$. (Hint: consider kinetic energy and rotation.)

Imagine a particle of mass m at position (x,y) in the plane. Suppose we rotated the particle about the y-axis (in 3D space) at some constant rotational velocity ω (for instance, let's measure ω in radians per second). What is the particle's kinetic energy?

Kinetic energy is half the product of mass and velocity squared, $K=\frac{1}{2}mv^2$. Here, the velocity is given by the product of the rotational velocity and the radius of the rotation. The rotational velocity is ω and the distance from (x,y) to the y-axis is $|x|$. That is, $v=|x|\omega$, so $K=\frac{1}{2}mx^2\omega^2$. The mx^2 roughly appears as the integrand of the integral we wish to compute.

Now suppose we have a lamina occupying a region R in the xy-plane with density function $\rho(x,y)$. Suppose we start rotating the entire lamina about the y-axis (in 3D space) at constant rotational velocity ω. What is the kinetic energy of the lamina?

Break the lamina up into tiny pieces. The piece near point (x, y) will have mass approximately equal to $\rho(x, y)\, dA$, where dA is the area of the piece in the xy-plane. This piece will have distance approximately $|x|$ from the y-axis, so by the previous computation, the kinetic energy of this piece is

$$K = \frac{1}{2}\omega^2 x^2 \rho(x, y)\, dA.$$

To find the total kinetic energy of the lamina, we add up all of these contributions and take the limit as the pieces get arbitrarily small, so

$$K_{\text{total}} = \frac{1}{2}\omega^2 \iint_R x^2 \rho(x, y)\, dA.$$

Comparing this to the usual kinetic energy formula, the double integral piece is behaving somewhat like mass–the larger it is, the more energy it will take to rotate this lamina at this speed about the y-axis. It's actually given a name: the moment of inertia about the y-axis. ■