Math 126 C Challenge Problems/Solutions

Problems Posted 7/24/2013
Solutions Posted 7/28/2013

1. Let $f(x, y) = (x + iy)^2$. Show that

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 2(x + iy).$$

(Identifying z with both (x, y) and $x + iy$, we may write $f(z) = z^2$, and the right-hand side of the above is $2z$.)

Since $(x + iy)^2 = (x^2 - y^2) + i(2xy)$, we compute

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}[(x^2 - y^2) + i(2xy)] - i\frac{\partial}{\partial y}[(x^2 - y^2) + i(2xy)]\right)\\ &= \frac{1}{2}\left([2x + i(2y)] - i[-2y + i(2x)]\right)\\ &= \frac{1}{2}\left(2x + i(2y) + i(2y) + 2x\right)\\ &= 2(x + iy).\end{aligned}$$

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2. Verify the complex "power rule":

$$\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x + iy)^n = n(x + iy)^{n-1},$$

(Using z as above and defining $\frac{d}{dz} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, this power rule becomes $\frac{d}{dz}z^n = nz^{n-1}$. This $\frac{d}{dz}$ is sometimes called a Wirtinger derivative.)

We could expand out $(x+iy)^n$ using the binomial theorem and use essentially the same derivation as above, but that gets messy. Let's instead use the chain rule (which holds for partial derivatives of complex-valued functions):

$$\begin{aligned}\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(x + iy)^n &= \frac{1}{2}\left(\frac{\partial}{\partial x}(x + iy)^n - i\frac{\partial}{\partial y}(x + iy)^n\right)\\ &= \frac{1}{2}\left(n(x + iy)^{n-1}\frac{\partial}{\partial x}(x + iy) - in(x + iy)^{n-1}\frac{\partial}{\partial y}(x + iy)\right)\\ &= \frac{1}{2}\left(n(x + iy)^{n-1} - in(x + iy)^{n-1}i\right)\\ &= n(x + iy)^{n-1}.\end{aligned}$$

A more standard approach to these ideas is to use complex differentiation: define $\frac{df}{dz}(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$, if it exists, where h is allowed to be a complex number. One can show that if the above limit exists for every z_0, then the result agrees with what you get by computing $\frac{d}{dz}f$ using the Wirtinger derivative above. One of the main ingredients of the equivalence is the Cauchy-Riemann equations from complex analysis, but they are as usual is well beyond the scope of this course. ■