Math 126 C Challenge Problems/Solutions

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1. Using July 12th's Challenge Problem 1, compute the two square roots of a complex number $x+iy$ in terms of polar coordinates. Define a function giving one of them; what is its domain?

A square root $u+iv$ of $x+iy$ by definition satisfies $(u+iv)^2 = x+iy$. Let's look at this in terms of polar coordinates. Let (r,θ) be the polar coordinates of $x+iy$ and let (r',θ') be the polar coordinates of $u+iv$. From the referenced challenge problem, the square of $u+iv$ has polar coordinates $(r'^2, 2\theta')$. Thus we require (r,θ) and $(r'^2, 2\theta')$ to represent the same Cartesian point. Since $r, r' \geq 0$, we must have $r' = \sqrt{r}$ (this is the usual square root of a non-negative number) and $2\theta' = \theta + 2\pi k$ for some integer k. That is, $\theta' = \theta/2 + \pi k$. Only $k=0$ and $k=1$ give different Cartesian points, so we have two candidates for (r',θ'):

$$(\sqrt{r}, \theta/2), \qquad (\sqrt{r}, \theta/2+\pi).$$

Indeed, these are distinct: one is on the opposite side of the other, so long as $r \neq 0$; if $r=0$, $x+iy = 0+0i$ has a single complex square root, namely itself.

Let's define $f(x,y)$ to give $u+iv$ corresponding to the first choice above, $(\sqrt{r}, \theta/2)$. This makes sense: given (x,y), we may compute its polar coordinates as (r,θ), compute $(\sqrt{r},\theta/2)$, compute the Cartesian point that represents, and convert that to the complex number $u+iv$. But wait: what about -1? If we say $-1+0i$ has polar coordinates $(1,\pi)$, then the square root has polar coordinates $(1,\pi/2)$, which is $0+i=i$. However, if we say $-1+0i$ has polar coordinates $(1,-\pi)$, then the square root has polar coordinates $(1,-\pi/2)$, which is $0-i=-i$. What went wrong?

In computing (r,θ), we had many choices of θ. Often it does not matter which θ we pick, but for this application it does matter. To fix the problem, we can just say we'll pick θ in the range $-\pi < \theta \leq \pi$. The above procedure then defines a complex square root function for each complex number, so the domain is the entire plane. ∎

2. Is it possible to define a continuous square root function on the entire complex plane?

The complex square root function defined above is defined in the entire complex plane, but it is not continuous. For points $-1+\epsilon i$ with $\epsilon > 0$ small, θ is slightly under π, so the square root has angle very close to $\pi/2$, hence the square root is very close to i. However, for points $-1-\epsilon i$ with $\epsilon > 0$ small, θ is slightly above $-\pi$, so the square root has angle very close to $-\pi/2$, hence the square root is very close to $-i$.

That is, we can find points near -1 which this function takes to very different values, so it is not continuous there. Indeed we can do the same for any negative real number–there's a whole line of discontinuities. Is it possible to fix this problem?

No. An intuitive sketch of an argument that can be made rigorous with a little real analysis is the following. At -1, you have to pick either i or $-i$ as the square root. Picking $-i$ forces you by continuity to pick square roots near $-i$ for points near -1. Now travel counter clockwise around the unit circle–you'll be forced to pick a square root with polar angle $-\pi/4$ for $-i$, polar angle 0 for 1, polar angle $\pi/4$ for i, and $\pi/2$ for -1. But we already said the square root of -1 was $-i$, which has polar angle $-\pi/2$, not $\pi/2$, so by the time we wrap all the way around we've added π to the original polar angle, which must result in a discontinuity. ∎