Math 126 C Challenge Problems/Solutions

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1. (Easy.) Compute the curvature at $\langle 1,0,0\rangle$ of the intersection of the sphere $x^2+y^2+z^2=1$ and the plane $y\cos(\theta)+z\sin(\theta)=0$ for fixed θ.

The planes in question pass through the origin, so they intersect the sphere in a circle of radius 1. As mentioned in class, such circles have constant curvature 1. ∎

2. Repeat (1) with the cylinder $x^2+y^2=1$.

At $\theta=0$, the normal vector is $\langle 0,1,0\rangle$, so the plane is perpendicular to the y-axis, hence it intersects this cylinder in a straight line, which has curvature 0. At $\theta=\pi/2$, the normal vector is $\langle 0,0,1\rangle$, so the plane is perpendicular to the z-axis, i.e. the plane is the xy-plane, which intersects the cylinder in a circle of radius 1, so the curvature is 1. Other angles give intermediate values.

In elementary differential geometry, these are the *principal curvatures* of the cylinder at this point. A complete definition is well beyond the scope of this course, but it turns out the product of the principal curvatures, called the *Gaussian curvature*, is an important invariant. Specifically, deforming a surface so that curve lengths do not change preserves Gaussian curvature. The most famous use of this idea is the following: can one ever have a faithful map of the Earth, that is, one which does not distort distances? The answer is no: the Gaussian curvature of a sphere is 1, while the Gaussian curvature of a plane is 0. If there were a faithful map, it would have to be obtained by distorting (part of) a sphere without changing distances, hence without changing Gaussian curvature. This result is sometimes called Gauss's "Theorema Egregium".

As for the original question, we first parameterize the intersection, and then use standard formulas to compute the curvature. Start with a vector $\langle\cos\alpha,\sin\alpha,0\rangle$ in the xy-plane. Imagine rotating it $\theta+\pi/2$ units about the x-axis. This operation is chosen so that the new vector will lie in the plane $y\cos\theta+z\sin\theta=0$. To do it, first compute the distance of $\langle\cos\alpha,\sin\alpha,0\rangle$ from the x-axis–this is simply $\sin\alpha$. Drawing out the situation, the new z-height will be this distance times $\sin(\theta+\pi/2)=\cos\theta$. The new x-position will be the same as the old one. The new y-position, however, will change: it will be the old distance to the x-axis times $\cos(\theta+\pi/2)=-\sin\theta$. In all, our new vector is

$$\langle\cos\alpha,-\sin\alpha\sin\theta,\sin\alpha\cos\theta\rangle.$$

(This reasoning is very similar to that used in deriving spherical coordinate/ Cartesian coordinate conversions.)

The ray through this vector lies on the plane, so we need to determine where it intersects the cylinder. That is, for fixed α and θ, for what $r>0$ is $r\langle\cos\alpha,-\sin\alpha\sin\theta,\sin\alpha\cos\theta\rangle$ on $x^2+y^2=1$? Plugging in and simplifying, we find

$$r=\frac{1}{\sqrt{\cos^2\alpha+\sin^2\alpha\sin^2\theta}}.$$

In all, our parameterization is

$$\mathbf{r}(\alpha)=\frac{\langle\cos\alpha,-\sin\alpha\sin\theta,\sin\alpha\cos\theta\rangle}{\sqrt{\cos^2\alpha+\sin^2\alpha\sin^2\theta}}.$$

Note that at $\theta = 0$, this gives $\mathbf{r}(\alpha) = \langle 1, 0, \tan\alpha\cos\theta\rangle$, which is a (non-constant-speed) parameterization of a line. Similarly at $\theta = \pi/2$, this gives $\mathbf{r}(\alpha) = \langle \cos\alpha, -\sin\alpha, 0\rangle$, which is a circle in the xy-plane. Both of these agree with the above analysis.

After a very lengthy computation one finds

$$\mathbf{r}'(\alpha) = \frac{\langle -\sin\alpha\sin^2\theta, -\cos\alpha\sin\theta, \cos\alpha\cos\theta\rangle}{(\cos^2\alpha + \sin^2\alpha\sin^2\theta)^{3/2}}$$

and, if $\mathbf{r}''(t) = \langle x''(t), y''(t), z''(t)\rangle$,

$$\begin{aligned}
x''(t) &= \frac{16(-3 + \cos(2\alpha) - 2\cos(2\theta)\sin^2\alpha)\cos\alpha\sin^2\theta}{(3 + \cos(2\alpha) - 2\cos(2\theta)\sin^2\alpha)^{5/2}} \\
y''(t) &= \frac{-(\cos(2\alpha)\cos^2\theta + \cos(2\theta))\sin\alpha\sin\theta)}{(\cos^2\alpha + \sin^2\alpha\sin^2\theta)^{5/2}} \\
z''(t) &= \frac{(\cos(2\alpha)\cos^2\theta + \cos(2\theta))\sin\alpha\cos\theta}{(\cos^2\alpha + \sin^2\alpha\sin^2\theta)^{5/2}}.
\end{aligned}$$

Conveniently, we want $\alpha = 0$, since this gives $\mathbf{r}(0) = \langle 1, 0, 0\rangle$. The above formulas vastly simplify here:

$$\begin{aligned}
\mathbf{r}'(0) &= \langle 0, -\sin\theta, \cos\theta\rangle \\
\mathbf{r}''(0) &= \langle -\sin^2\theta, 0, 0\rangle.
\end{aligned}$$

(Indeed, we could have avoided computing $y''(t)$ and $z''(t)$ by noting that physically acceleration should be parallel to the x-axis on this curve at $\langle 1, 0, 0\rangle$.) Note that $|\mathbf{r}'(0)| = 1$. We also have $\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 0, -\cos t\sin^2 t, -\sin^3 t\rangle$, hence

$$\kappa(\theta) = |\mathbf{r}'(0) \times \mathbf{r}''(0)| = |\sin^2\theta|.$$

At $\theta = 0$, this is 0, and at $\theta = \pi/2$, this is 1, in agreement with the above. ∎