Math 126 C Challenge Problems/Solutions

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1. (Easy) We can "multiply" 2D vectors by identifying them with complex numbers: $\langle x, y\rangle \equiv x + iy$. Use 6/28 Challenge Problems to interpret complex number multiplication in terms of polar coordinates for the corresponding vectors.

6/28 Challenge Problems showed that the magnitude of the product of complex numbers is the product of their magnitudes, and the angle the product makes with the positive x-axis measured counterclockwise is the sum of the angles. That is, if we identify $x_1 + iy_1$ with polar coordinates (r_1, θ_1) and $x_2 + iy_2$ with polar coordinates (r_2, θ_2), then the product $(x_1 + iy_1)(x_2 + iy_2)$ has polar coordinates $(r_1 r_2, \theta_1 + \theta_2)$.

This allows us to solve certain problems very quickly that would otherwise take some time. For instance, when is

$$(x + iy)^3 = (x - iy)?$$

Note that we can multiply both sides by $x + iy$, and $(x - iy)(x + iy) = x^2 + y^2$, so $(x + iy)^4 = x^2 + y^2 \geq 0$ is real. Thinking of $x + iy$ as (r, θ), we see $(x + iy)^4$ is $(r^4, 4\theta)$. In polar, $x^2 + y^2$ is $(r^2, 0)$. For $(r^4, 4\theta) = (r^2, \theta)$, we require $[r = 0]$ or $[r = 1$ and $\theta = 0, \pi/2, \pi, 3\pi/2]$. These correspond to complex numbers 0, $1, i, -1$, and $-i$. ∎

2. Show that a curve $\mathbf{r}(t)$ whose tangent vectors at each point are perpendicular to $\mathbf{r}(t)$ is contained in a sphere centered at the origin.

We have $\mathbf{r}'(t)$ perpendicular to $\mathbf{r}(t)$, i.e. $\mathbf{r}'(t) \cdot \mathbf{r}(t)$. But we can use the product rule for dot products to compute $d/dt(|\mathbf{r}(t)|^2) = d/dt(\mathbf{r}(t) \cdot \mathbf{r}(t)) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, so $|\mathbf{r}(t)|$ is constant. That is, $\mathbf{r}(t)$ is contained in a sphere centered at the origin. ∎