Math 126 C Challenge Problems/Solutions

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1. In three dimensional space...

(a) Compute the (smallest) distance between a line and a point.

(b) Compute the (smallest) distance between two lines.

(a) Say the point is (x, y, z) and the line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ as usual. The distance between $\mathbf{P} = (x, y, z)$ and $\mathbf{r}(t)$ is a function of one variable, t, namely $|\mathbf{P} - \mathbf{r}(t)|$, which we can minimize using standard 1-variable calculus. It's easier to minimize the square of the distance, which is $D^2 = (\mathbf{P} - \mathbf{r}(t)) \cdot (\mathbf{P} - \mathbf{r}(t))$. Using linearity and differentiation rules for the dot product, we have

$$\begin{aligned}(D^2)' &= (\mathbf{P} \cdot \mathbf{P} - 2\mathbf{P} \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}(t))' \\ &= 0 - 2\mathbf{P} \cdot \mathbf{r}'(t) + 2\mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= -2\mathbf{P} \cdot \mathbf{v} + 2\mathbf{r}(t) \cdot \mathbf{v} \\ &= -2\mathbf{P} \cdot \mathbf{v} + 2\mathbf{r}_0 \cdot \mathbf{v} + 2t\mathbf{v} \cdot \mathbf{v}. \qquad (1)\end{aligned}$$

Since $\mathbf{v} \neq 0$ (otherwise $\mathbf{r}$ would just trace out a point rather than a line), we can set (**??**) equal to zero, divide by $|\mathbf{v}|^2$, and solve for t to get $t_0 = (\mathbf{P} - \mathbf{r}_0) \cdot \mathbf{v}/|\mathbf{v}|^2$. Plugging this in gives the minimum distance as

$$\begin{aligned}D(t_0) &= |(\mathbf{P} - \mathbf{r}_0) - ((\mathbf{P} - \mathbf{r}_0) \cdot \mathbf{v}/|\mathbf{v}|^2)\mathbf{v}| \\ &= |(\mathbf{P} - \mathbf{r}_0) - \mathrm{proj}_{\mathbf{v}}(\mathbf{P} - \mathbf{r}_0)|.\end{aligned}$$

We could have arrived at the same formula much earlier geometrically: look at the vector from $\mathbf{P}$ to $\mathbf{r}_0$; it has two components, one directly from $\mathbf{P}$ to the line, and one along $\mathbf{v}$; subtract off the component along $\mathbf{v}$, which is by definition the projection of $\mathbf{P} - \mathbf{r}_0$ along $\mathbf{v}$; compute the length of the remainder.

(b) Suppose the two lines are $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ and $\mathbf{s}(u) = \mathbf{s}_0 + u\mathbf{w}$. We could mimic the calculus in (a) as follows: fix u and let $\mathbf{P} = \mathbf{s}(u)$; use the formula above for the minimum distance between $\mathbf{P}$ and $\mathbf{r}(t)$; now consider this formula as a function of s, and minimize it over all s again using 1-variable calculus.

Alternatively, there is another short geometric argument. The shortest line segment from $\mathbf{r}$ to $\mathbf{s}$ must make a right angle with both lines. Suppose this segment has direction vector $\mathbf{d}$; then $\mathbf{d}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, so we can take $\mathbf{d} = \mathbf{v} \times \mathbf{w}$—unless $\mathbf{v}$ and $\mathbf{w}$ are parallel. If $\mathbf{v}$ and $\mathbf{w}$ are parallel, we can just use the result of (a) to compute the distance between them (why?), so suppose this is not the case. Consider the vector from $\mathbf{r}_0$ to $\mathbf{s}_0$. The distance from the first line to the second is given by projecting $\mathbf{r}_0 - \mathbf{s}_0$ onto $\mathbf{d}$ (why?), i.e. the final answer is

$$D = |\mathrm{proj}_{\mathbf{v} \times \mathbf{w}}(\mathbf{r}_0 - \mathbf{s}_0)|.$$

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2. (Hard.) Extend your ideas in (1) to compute the distance between a pair of two-dimensional planes in four-dimensional space. (You'll probably have to figure out how to represent such a plane first.)

Let's represent such planes as follows:

$$\mathbf{r}(t_1, t_2) = \mathbf{r}_0 + t_1\mathbf{v_1} + t_2\mathbf{v_2}$$
$$\mathbf{s}(u_1, u_2) = \mathbf{s}_0 + u_1\mathbf{w_1} + u_2\mathbf{w_2}$$

where all the vectors involved are 4-dimensional. We require $\mathbf{v}_1$ and $\mathbf{v}_2$ to be non-zero, non-parallel vectors, since otherwise $\mathbf{r}$ doesn't actually trace out a plane. One way to solve this problem is to use a blend of calculus and geometric reasoning. Since the details are long and involved, I'll only sketch the argument.

Fix a point $\mathbf{P}$ on $\mathbf{s}$'s plane. Compute the minimum distance between $\mathbf{P}$ and $\mathbf{r}(t_1, c)$ where c is a fixed constant and t_1 is varying; this calculation is very similar to that in 1(a). After that, compute the minimum of the minimums by allowing c to vary—this will give a formula for the minimum distance between $\mathbf{P}$ and the $\mathbf{r}$-plane. Repeat this two more times to get a formula for the distance between the two planes.

Note: I have not seen a full solution to this problem. Please let me know if you find one or if you devise a better solution technique and I will update this writeup accordingly. My hope in asking the question was that someone would come up with a creative, clean solution. ∎