Math 126 C Challenge Problems/Solutions

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1. Compute the volume of the paralellpiped spanned by $\langle 1, a, a^2\rangle$, $\langle 1, b, b^2\rangle$, $\langle 1, c, c^2\rangle$. Write your answer as a product of linear factors. (To be continued.)

Using the scalar triple product, we have the (signed) volume as

$$\begin{aligned} V &= \langle 1, a, a^2\rangle \cdot (\langle 1, b, b^2\rangle \times \langle 1, c, c^2\rangle) \\ &= \langle 1, a, a^2\rangle \cdot [(bc^2 - cb^2)\mathbf{i} - (c^2 - b^2)\mathbf{j} + (c-b)\mathbf{k}] \\ &= \langle 1, a, a^2\rangle \cdot [bc(c-b)\mathbf{i} - (c-b)(c+b)\mathbf{j} + (c-b)\mathbf{k}] \\ &= (c-b)(bc - (c+b)a + a^2) \\ &= (c-b)(b-a)(c-a) \\ &= (a-b)(b-c)(c-a). \end{aligned}$$

This is called a *Vandermonde determinant*. The same type of formula holds in higher dimensions for "parallelotopes", though the general proof is significantly more involved. ∎

2. The cross product is not generally associative, i.e. it is not generally true that $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w}$. However, show that

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) + \mathbf{v}\times(\mathbf{w}\times\mathbf{u}) + \mathbf{w}\times(\mathbf{u}\times\mathbf{v}) = \mathbf{0}.$$

[Equivalently,

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\times\mathbf{w} - \mathbf{v}\times(\mathbf{w}\times\mathbf{u}),$$

so in a sense the term $-\mathbf{v}\times(\mathbf{w}\times\mathbf{u})$ measures the failure of associativity precisely.]

[Hint: see Exercise 12.4.51 of Stewart.]

There are several different proofs. Stewart suggests one based on the identity

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}.$$

Applying this identity three times, you'll find the terms cancel in pairs. The identity can be proved directly from the definitions, or you can try finding a more clever proof.

Here is a very different proof, where I'll assume a small amount of linear algebra, specifically knowledge of matrix multiplication. First, an apparent digression. Let A, B, and C be square matrices of the same size. Define an operation $[A, B] = AB - BA$, called the *commutator* of A and B. Note that $[A, B] = 0$ if and only if $AB = BA$, so the commutator in a sense measures the failure of commutativity precisely. The commutator is not generally associative, i.e. $[A, [B, C]] \neq [[A, B], C]$, however you can quickly check that the commutator does obey the Jacobi identity, just by expanding the terms out and noting cancellations:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

(Here 0 denotes the zero matrix, i.e. a matrix with 0's in each entry.) Next, we'll find a way to turn 3D vectors into particular 3×3 matrices in a very nice way: the $\mathbf{0}$ vector is sent to the 0 matrix, the "translation" can be reversed, and cross products are turned into commutators. Specifically, given $\mathbf{v} = \langle x, y, z \rangle$, the associated matrix is

$$\begin{pmatrix} 0 & -x & -y \\ x & 0 & -z \\ y & z & 0 \end{pmatrix}.$$

Obviously $\mathbf{0}$ is sent to the zero matrix and given any matrix of this form we can figure out the vector which gave it birth so the operation can be reversed. Finally, suppose $A_{\mathbf{u}}$ is the matrix associated with the vector $\mathbf{u}$ and $B_{\mathbf{v}}$ with $\mathbf{v}$. One may check directly by multiplying the matrices and applying the definitions that the matrix associated with $\mathbf{u} \times \mathbf{v}$ is precisely $[A_{\mathbf{u}}, B_{\mathbf{v}}]$. We can now apply the Jacobi identity for 3×3 matrices to give the Jacobi identity for cross products, as follows; $\rightarrow$ denotes translation to or from the matrix form.

$$\begin{aligned} &\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \\ &\quad \rightarrow [A_{\mathbf{u}}, [B_{\mathbf{v}}, C_{\mathbf{w}}]] + [B_{\mathbf{v}}, [C_{\mathbf{w}}, A_{\mathbf{u}}]] + [C_{\mathbf{w}}, [A_{\mathbf{u}}, B_{\mathbf{v}}]] \\ &\quad = 0 \\ &\quad \rightarrow \mathbf{0}. \end{aligned}$$

This reasoning essentially shows that the cross product makes the set of 3D vectors into what is called a *Lie algebra*. A key property of Lie algebras is that they satisfy the Jacobi identity. The theory of Lie algebras is rich and interesting, but of course also well beyond the scope of this course. ■