Math 126 C Challenge Problems/Solutions

Problems Posted 6/28/2013 Solutions Posted 7/01/2013

1. Given a complex number $a+bi$, form a vector $\langle a,b\rangle$. We call the length of $\langle a,b\rangle$ the **magnitude of** $a+bi$, which we write as $|a+bi|$. Show that $|wz|=|w||z|$ for any complex numbers w,z.

Write $w=a+bi$, $z=c+di$. We have $wz=(a+bi)(c+di)=(ac-bd)+(ad+bc)i$. We may now compute $|wz|$ directly:

$$\begin{aligned}|wz| &= |\langle ac-bd, ad+bc\rangle| \\ &= \sqrt{(ac-bd)^2+(ad+bc)^2} \\ &= \sqrt{a^2c^2-2acbd+b^2d^2+a^2d^2+2adbc+b^2c^2} \\ &= \sqrt{a^2(c^2+d^2)+b^2(d^2+c^2)} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= |\langle a,b\rangle||\langle c,d\rangle| \\ &= |w||z|.\end{aligned}$$

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2. If v is any complex number, write θ_v for the angle v makes with the positive x-axis measured counterclockwise. Show that $\theta_{wz}=\theta_w+\theta_z \pmod{2\pi}$ for any complex numbers w,z.

Identify complex numbers and vectors as in (1), so we may take dot products of complex numbers. Note that $v\cdot(1+0i)=|v|\cos(\theta_v)$. Indeed, if we write $w=a+bi$, $z=c+di$, we have

$$\begin{aligned}a = w\cdot 1 &= |w|\cos(\theta_w), \\ c = z\cdot 1 &= |z|\cos(\theta_z), \\ ac-bd = (wz)\cdot 1 &= |wz|\cos(\theta_{wz}).\end{aligned}$$

Substituting the first two lines into the third and using the result from (1) gives

$$|w|\cos(\theta_w)|z|\cos(\theta_z)-bd=|w||z|\cos(\theta_{wz}).$$

b and d also have nice interpretations, namely $b=|w|\sin(\theta_w)$ and $d=|z|\sin(\theta_z)$. You can see this for w by drawing out the triangle with vertices at $\langle 0,0\rangle, \langle a,0\rangle, \langle a,b\rangle$. Note that the counterclockwise convention is used in completely justifying this step. We now have

$$|w||z|\cos(\theta_w)\cos(\theta_z)-|w||z|\sin(\theta_w)\sin(\theta_z)=|w||z|\cos(\theta_{wz}).$$

Cancel the $|w||z|$'s and note the standard trig identity

$$\cos(\theta_w)\cos(\theta_z)-\sin(\theta_w)\sin(\theta_z)=\cos(\theta_w+\theta_z)$$

to get $\cos(\theta_w+\theta_z)=\cos(\theta_{wz})$. We may repeat this process using $v\cdot(0+1i)=|v|\sin(\theta_v)$ instead and arrive also at $\sin(\theta_w+\theta_z)=\sin(\theta_{wz})$, which together with the above cos version gives $\theta_w+\theta_z=\theta_{wz} \pmod{2\pi}$. ■