Math 126 C Challenge Problems/Solutions
Problems Posted 6/26/2013
Solutions Posted 6/28/2013

---

1. Given $|\mathbf{v} + \mathbf{w}|$ and $|\mathbf{v} - \mathbf{w}|$, compute $\mathbf{v} \cdot \mathbf{w}$.

Using the relationship between the magnitude and the dot product and the linearity of the dot product (see §12.3, Theorem 2), we have

$$|\mathbf{v} + \mathbf{w}|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})$$
$$= \mathbf{v} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{w}$$
$$= |\mathbf{v}|^2 + 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2.$$

In the same way we find

$$|\mathbf{v} - \mathbf{w}|^2 = |\mathbf{v}|^2 - 2(\mathbf{v} \cdot \mathbf{w}) + |\mathbf{w}|^2.$$

Thus subtracting the second from the first gives $4(\mathbf{v} \cdot \mathbf{w})$, so that

$$\mathbf{v} \cdot \mathbf{w} = \frac{1}{4}(|\mathbf{v} + \mathbf{w}|^2 - |\mathbf{v} - \mathbf{w}|^2).$$

This is the so-called "polarization identity" (for Euclidean spaces; there are more general versions). It's perhaps a bit surprising that we can compute the dot product function given only a way to compute the lengths of arbitrary vectors. In more general settings (Banach spaces), this relationship breaks down and provides for a potentially richer variety of objects to study. ∎

---

2. Find a non-zero vector perpendicular to both $\langle 0, 1, 2 \rangle$ and $\langle 3, 4, 5 \rangle$. Devise a general strategy for solving this type of problem.

Suppose such a vector is $\langle x, y, z \rangle$. The dot product of this with the given vectors must be zero, so we may solve the following system of equations:

$$\langle x, y, z \rangle \cdot \langle 0, 1, 2 \rangle = y + 2z = 0$$
$$\langle x, y, z \rangle \cdot \langle 3, 4, 5 \rangle = 3x + 4y + 5z = 0$$

An obvious solution to the first equation is $y = 1, z = -1$. This forces $x = 2$ in the second equation, so $\langle 2, 1, -1 \rangle$ is a solution. In general there are only two equations and three unknowns, so, except for degenerate cases where one of the input vectors is $\mathbf{0}$, there are always (infinitely many) solutions.

Another technique: say the input vectors are $\mathbf{u}, \mathbf{v}$. Pick any vector $\mathbf{w}$ not in the plane they span; there are several ways to do this, though I'll leave that to you. Now compute

$$\mathbf{w}' = \mathbf{w} - \text{proj}_{\mathbf{u}}\mathbf{w} - \text{proj}_{\mathbf{v}}\mathbf{w}.$$

Since we subtract off the portions of $\mathbf{w}$ in the direction of the input vectors, the result is perpendicular to each. Since $\mathbf{w}$ was not in the plane they span, $\mathbf{w}'$ is non-zero, as needed. This operation is the key idea in the "Gram-Schmidt Orthogonalization" process.

Also, cross products solve this problem immediately, though they don't work in general dimensions whereas the above methods do. ∎