Math 126 C Challenge Problems/Solutions

Problems Posted 6/24/2013
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1. Let $\mathbf{v} = \langle a, b \rangle$, $\mathbf{w} = \langle c, d \rangle$. These two vectors determine a parallelogram with vertices at the origin, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$. Compute the area of this parallelogram in terms of a, b, c, d.

The parallelogram has vertices at $(0,0), (a,b), (c,d), (a+c, b+d)$. Move triangles around as in the following diagram to create a rectangle:

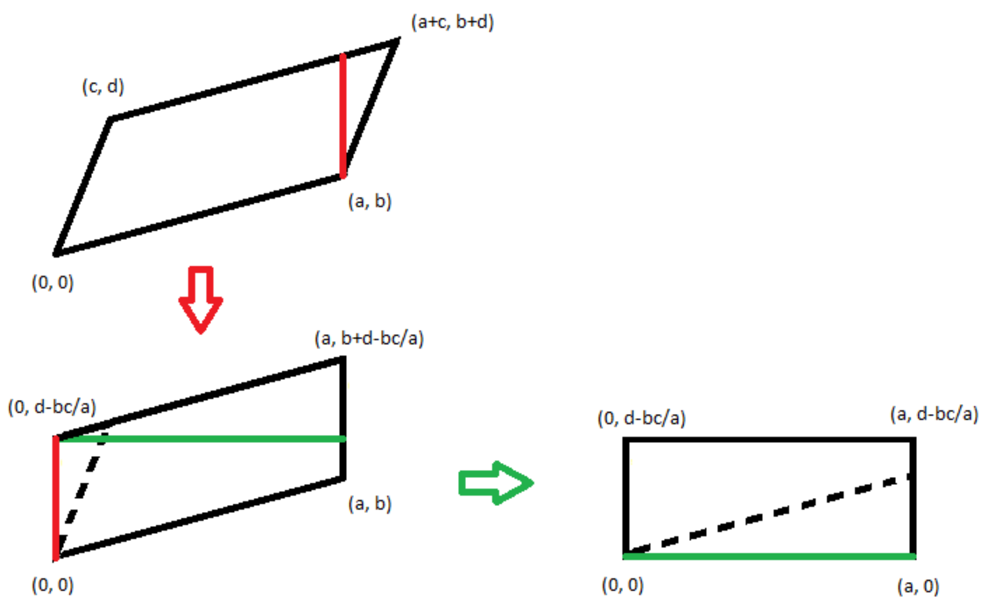

Figure 1: Computing the area of a parallelogram.

Thus the area is $a(d - bc/a) = ad - bc$. (One may verify the formula still works if $a = 0$.) Strictly speaking, this is the *signed area*, since it might be negative. The sign comes from the *orientation* of the parallelogram, which is somewhat involved to define rigorously. Intuitively, $\langle 1, 0 \rangle, \langle 0, 1 \rangle$ gives a parellelogram with opposite orientation to $\langle 0, 1 \rangle, \langle 1, 0 \rangle$; indeed, the area formula above gives 1 for the first and -1 for the second.

Note: the area formula is the same as the determinant of $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. ■

$\boxed{\text{2. Compute the area determined by the vectors } \mathbf{v} \text{ and } \mathbf{v}-\mathbf{w}\text{. Explain the result.}}$

Use the area formula from (1):

$$\begin{aligned} \text{Area}(\mathbf{v}, \mathbf{v}-\mathbf{w}) &= \text{Area}(\langle a, b\rangle, \langle a-c, b-d\rangle) \\ &= a(b-d) - b(a-c) \\ &= -(ad-bc) \\ &= -\text{Area}(\mathbf{v}, \mathbf{w}). \end{aligned}$$

Adding a multiple of the first vector to the second vector just "skews" the resulting parallelogram without changing its area. The final negative sign comes about since negating $\mathbf{w}$ switches the orientation of the resulting parallelogram. ∎