Math 126 C Worksheet Solutions

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Note: Send corrections, if any, to `jps314@math.washington.edu`.

(1) We compute

$$g_x = \frac{1}{y+3z} \qquad g_y = -\frac{x}{(y+3z)^2} \qquad g_z = -\frac{3x}{(y+3z)^2}$$

$$\Rightarrow g_{xx} = 0 \qquad \Rightarrow g_{yy} = \frac{2x}{(y+3z)^3} \qquad \Rightarrow g_{zz} = \frac{18x}{(y+3z)^2}$$

$$g_{xy} = -\frac{1}{(y+3z)^2} \qquad g_{yz} = \frac{6x}{(y+3z)^3}$$

$$g_{xz} = -\frac{3}{(y+3z)^2}$$

Since $g_{xy} = g_{yx}$, $g_{xz} = g_{zx}$, and $g_{yz} = g_{zy}$ here, this is all $3 \times 3 = 9$ second partials.

(2) We compute

$$\begin{aligned} f_x &= 2y\cos(2x-y) \\ f_y &= -y\cos(2x-y) + \sin(2x-y), \end{aligned}$$

so the linearization at $(1,2)$ is

$$\begin{aligned} L(x,y) &= f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) \\ &= 4(x-1) - 2(y-2). \end{aligned}$$

The approximation is then

$$f(1.02, 1.9) \approx L(1.02, 1.9) = 4(0.02) - 2(-0.1) = \frac{7}{25}.$$

<u>Note:</u> Here, $f(1,2) = 0$. In general it's easy to forget to add $f(a,b)$. For instance, at $(1,2)$, $df = 4\,dx - 2\,dy$, so $df = 7/25$, but this is not really the approximation for $f(1.02, 1.9)$—that is given by $f(1.02, 1.9) = f(1,2) + df = 0 + 7/25 = 7/25$.

(3) No single number should be favored over the others, so intuitively $4+4+4 = 12$ should be the solution. More rigorously, we have $x+y+z = 12$ and we want to minimize $C(x,y,z) = x^2+y^2+z^2$. The variables in the C function are not independent; solve for x in terms of y and z to reduce C to a function of two independent variables, which we can then extremize as usual. We find $x = 12 - y - z$ so

$$C(y,z) = (12-y-z)^2 + y^2 + z^2.$$

Find critical points:

$$\begin{aligned} C_y &= -2(12-y-z) + 2y = 0 \\ C_z &= -2(12-y-z) + 2z = 0. \end{aligned}$$

This is a (nondegenerate) linear system in two variables, so it has a unique solution. Plugging in $y = z = 4$ directly works, so that must be the solution. (Alternatively you can solve the system as usual and get $y = z = 4$.) So, the unique critical point is at $y = z = 4$, forcing $x = 4$. This is a minimum from the Second Derivatives Test: $C_{yy} = 4 = C_{zz}$, $C_{yz} = 2 = C_{zy}$, so $D = 16 - 4 = 10 > 0$ and $C_{yy} > 0$.

Strictly speaking we should check the boundaries of the region of points under consideration (namely, the positive y and z axes, and the behavior as y and z go off to infinity). But, the question seems to just want us to do the Second Derivatives Test part of the verification, so we stop here.

(4) a

- First way:

$$\begin{aligned}\iint_R \frac{x}{1+xy}\,dA &= \int_0^1 \int_0^2 \frac{x}{1+xy}\,dy\,dx \\ &= \int_0^1 \ln(1+xy)\big|_{y=0}^{2}\,dx \\ &= \int_0^1 \ln(1+2x)\,dx.\end{aligned}$$

  In general one can integrate the logarithm using integration by parts. Here, $u = \ln(1+2x)$, $dv = dx$, so $du = 2/(1+2x)$ and $v = x$. Now

$$\begin{aligned}\int_0^1 \ln(1+2x)\,dx &= \ln(1+2x)x|_0^1 - \int_0^1 \frac{2x}{1+2x}\,dx \\ &= \ln(3) - \int_0^1 \left(1 - \frac{1}{1+2x}\right) dx \\ &= \ln(3) - 1 + \int_0^1 \frac{1}{1+2x}\,dx \\ &= \ln(3) - 1 + \left.\frac{\ln(1+2x)}{2}\right|_0^1 \\ &= \ln(3) - 1 + \frac{\ln(3)}{2} \\ &= \frac{3\ln(3)}{2} - 1.\end{aligned}$$

  (The second step could also be done with a u-substitution, $u = 1+2x$, since then $2x = u-1$.)

- The second way, $dx\,dy$, is more difficult and involves integrating $\ln(1+y)/y^2$, which uses repeated integration by parts. You're probably not intended to solve the problem this way since it would be too time-consuming and error-prone, so such a solution is not included here.

b It is very important to draw the triangle correctly. In words, it is the half of the rectangle $[0,1]\times[0,2]$ above the line $y = 2x$. Two solutions:

- $dx\,dy$: fix y and determine for which values of x the point (x,y) lies on the figure. To do so, draw a horizontal (horizontal since y, height, is fixed) line through the triangle. The x-values hit will be 0 on the left and on the right the x-value will satisfy $y = 2x$ since (x,y) lies on the line $y = 2x$. On the right, the x-value in terms of y is then $y/2$. y varies from 0 to 2, so we have

$$\begin{aligned}\int_0^2 \int_0^{y/2} xy^2\,dx\,dy &= \int_0^2 \left.\frac{x^2y^2}{2}\right|_{x=0}^{y/2} dy \\ &= \int_0^2 \frac{y^4}{8}\,dy = \left.\frac{y^5}{40}\right|_{y=0}^{2} \\ &= \frac{4}{5}.\end{aligned}$$

- $dy\,dx$: fix x and determine for which values of y the point (x,y) lies on the figure. To do so, draw a vertical (vertical since x is fixed) line through the triangle. The y-values hit will be 2 at the top and at the bottom the y-value will satisfy $y = 2x$ since (x,y) lies on the line $y = 2x$. On the bottom, the y-value in terms of x is then $2x$. x varies from 0 to 1, so we have

$$\begin{aligned}
\int_0^1 \int_{2x}^2 xy^2\, dy\, dx &= \int_0^1 \left.\frac{xy^3}{3}\right|_{y=2x}^2 dx \\
&= \int_0^1 \frac{8x}{3} - \frac{8x^4}{3}\, dx = \left.\left(\frac{4x^2}{3} - \frac{8x^5}{15}\right)\right|_{x=0}^1 \\
&= \frac{4}{3} - \frac{8}{15} = \frac{4}{5}.
\end{aligned}$$

(5) We need to compute arc length, which requires computing the speed.

$$\begin{aligned}
|\mathbf{r}'(t)| &= |\langle e^t, 2e^t \sin(t) + 2e^t \cos(t), 2e^t \cos(t) - 2e^t \sin(t)\rangle| \\
&= |e^t \langle 1, 2\cos(t) + 2\sin(t), 2\cos(t) - 2\sin(t)\rangle| \\
&= e^t \sqrt{1 + (2\cos(t) + 2\sin(t))^2 + (2\cos(t) - 2\sin(t))^2} \\
&= e^t \sqrt{1 + 4\cos^2(t) + 8\cos(t)\sin(t) + 4\sin^2(t) + 4\cos^2(t) - 8\cos(t)\sin(t) + 4\sin^2(t)} \\
&= e^t \sqrt{1 + 4 + 4} \\
&= 3e^t.
\end{aligned}$$

The arc length $s(t)$ is measured from the point $(1, 0, 2)$, which corresponds to $t = 0$. So by definition of arc length, we have

$$\begin{aligned}
s(t) &= \int_0^t |\mathbf{r}(u)|\, du \\
&= \int_0^t 3e^u\, du \\
&= 3e^u\big|_0^t \\
&= 3e^t - 3 = 3(e^t - 1).
\end{aligned}$$

(The integral starts at 0 since the arc length is measured from a point corresponding to $t = 0$.) Solving for t in terms of s gives

$$\begin{aligned}
e^t &= \frac{s}{3} + 1 \\
&\Rightarrow t = \ln\left(\frac{s}{3} + 1\right).
\end{aligned}$$

Substituting this into the original function (and pulling out e^t from each term for convenience) gives

$$\mathbf{r}(s) = \left(\frac{s}{3} + 1\right)\left[\mathbf{i} + 2\sin\ln\left(\frac{s}{3} + 1\right)\mathbf{j} + 2\cos\ln\left(\frac{s}{3} + 1\right)\mathbf{k}\right].$$