Prof. Perkins Spring 2009 Math 126 Midterm 1 Solutions

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Note: Send corrections, if any, to `jps314@math.washington.edu`.

(1) Recall the formula for the slope of a polar curve,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Here $r = e^\theta$ implies $dr/d\theta = e^\theta$. Applying the formula and canceling the e^θ's gives a slope of

$$\frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}.$$

We need to know when this is zero, which occurs when the numerator is zero. (Note: if $\cos\theta = \sin\theta$ we divide by 0, but for these values the numerator is non-zero, so these are vertical tangent lines.) One computes $\sin\theta + \cos\theta = 0$ for $0 \le \theta \le 2\pi$ for $\boxed{\theta = 3\pi/4, 7\pi/4}$.

(2) Recall the slope formula for a parametric curve in the plane,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Also recall the second derivative formula for such a curve,

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

The curve is concave up if $\frac{d^2y}{dx^2} > 0$. Noting that $dx/dt = 2t, dy/dt = 3t^2 - 12$, one finds with the above formulas that

$$\frac{d^2y}{dx^2} = \frac{3t^2 + 12}{4t^3}.$$

The numerator is always positive. The denominator is positive for $t > 0$ and negative for $t < 0$; at $t = 0$, the curve has a vertical tangent line, and in any case is not concave up there. So, the curve is concave up precisely for $\boxed{t > 0}$.

(3) Recall the arc length formula,

$$\int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt.$$

The point $(0, 0)$ corresponds to $t = 0$, and $(2, 12)$ corresponds to $t = 2$ (check this for yourself). One computes

$$|\mathbf{r}'(t)| = 3(t^2 + 1)$$

so the integral above gives $\boxed{14}$.

(4) Several solutions. One is to rewrite the nearly symmetric equations in standard form and convert to vector form:

$$\frac{x-5}{-3} = \frac{y-4}{-5} = \frac{z+2}{7}, \qquad \frac{x-4}{-1} = \frac{y+7}{3} = \frac{z-3}{1}$$

$$\Rightarrow \mathbf{r}(t) = \langle 5, 4, -2\rangle + t\langle -3, -5, 7\rangle, \qquad \mathbf{s}(u) = \langle 4, -7, 3\rangle + u\langle -1, 3, 1\rangle.$$

Supposing for some t and u we have $\mathbf{r}(t) = \mathbf{s}(u)$ gives a system of three equations in two unknowns:

$$\begin{aligned} 5 - 3t &= 4 - u \\ 4 - 5t &= -7 + 3u \\ -2 + 7t &= 3 + u. \end{aligned}$$

Solving them gives a unique solution of $t = 1, u = 2$, corresponding to the point $\boxed{\mathbf{r}(1) = \mathbf{s}(2) = \langle 2, -1, 5\rangle}$.

(5) The point $(3, 0, 0)$ corresponds to $t = -1$; note that $t = 1$ corresponds to $(3, 0, \ln(5))$. Compute the direction vector of the tangent line:

$$\mathbf{v} = \mathbf{r}'(-1) = \left\langle \frac{t}{\sqrt{t^2+8}}, t\pi\cos(\pi t) + \sin(\pi t), \frac{2}{2t+3} \right\rangle\Bigg|_{t=-1} = \left\langle -\frac{1}{3}, \pi, 2 \right\rangle.$$

Since $(3, 0, 0)$ is on the line, it has vector form $\mathbf{L}(t) = \langle 3, 0, 0\rangle + t\left\langle -\frac{1}{3}, \pi, 2\right\rangle$. In parametric form, this is

$$\boxed{x(t) = 3 - \tfrac{t}{3}, \qquad y(t) = \pi t, \qquad z(t) = 2t.}$$

(6) The line of intersection has direction vector given by the cross product of the normals of the planes. These are $\langle 1, 0, 0\rangle$ and $\langle 0, 1, 1\rangle$, respectively, and their cross product is $\mathbf{v} = \langle 0, -1, 1\rangle$. We can find a point on the line of intersection by inspection—it must have x coordinate 3 and $y + z = 2$, so, for instance, $\mathbf{r}_0 = \langle 3, 1, 1\rangle$ works.

Now, how does one find the distance between a line and a point? I wrote up two methods for Challenge Problem 1(a) at `http://www.math.washington.edu/~jps314/m126/cp/cp0708.pdf`. The distance between a point $\mathbf{P}$ and a line with direction vector $\mathbf{v}$ with some point $\mathbf{r}_0$ on the line is then

$$|(\mathbf{P} - \mathbf{r}_0) - \mathrm{proj}_{\mathbf{v}}(\mathbf{P} - \mathbf{r}_0)|.$$

Here, with $\mathbf{P} = \langle 2, 1, -1\rangle$ given, we have $\mathbf{P} - \mathbf{r}_0 = \langle -1, 0, -2\rangle$ and one can compute the projection as $\langle 0, 1, -1\rangle$. The distance is then

$$|\langle -1, -1, -1\rangle| = \boxed{\sqrt{3}.}$$