

Math 126 Midterm 2 Review Sheet
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Note: Section numbers refer to Stewart, Multivariable Calculus, 7th Edition.

1 Parameterizations

1.1 TNB Frame, §13.3

$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ throughout. See Midterm 1 Review for further conventions.

- The *unit tangent vector* is $\mathbf{T}(t) = \text{Unit}(\mathbf{r}'(t)) = \mathbf{r}'(t)/|\mathbf{r}'(t)|$
- The *unit normal vector* is $\mathbf{N}(t) = \text{Unit}(\mathbf{T}'(t)) = \mathbf{T}'(t)/|\mathbf{T}'(t)|$.
- The *binormal vector* is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$
- The *osculating plane at t* is spanned by $\mathbf{T}(t)$ and $\mathbf{N}(t)$, or equivalently by $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. It has $\mathbf{B}(t)$ as normal vector.
- The *normal plane at t* is spanned by $\mathbf{N}(t)$ and $\mathbf{B}(t)$. It has $\mathbf{T}(t)$, or equivalently $\mathbf{r}'(t)$, as normal vector.
- The *osculating circle at t* is the circle in the osculating plane with radius $1/\kappa(t)$ (where $\kappa(t)$ is the curvature) and center $\mathbf{r}(t) + \mathbf{N}(t)/\kappa(t)$. The curvature of a circle of radius R is then (constantly) $1/R$.

1.2 Components of Acceleration, §13.4

Let $\mathbf{r}(t)$ be the path of a particle. The *velocity vector* is $\mathbf{r}'(t)$, the *speed* is $|\mathbf{r}'(t)|$, and the *acceleration vector* is $\mathbf{a}(t) = \mathbf{r}''(t)$. The acceleration vector describes both how much and in which direction the particle is accelerating at any given time. It happens that the acceleration vector is always in the plane containing the tangent and normal vectors, \mathbf{T} and \mathbf{N} , i.e. the osculating plane.

- The acceleration vector is given by

$$\mathbf{a} = \mathbf{r}'' = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

Note that $v' = |\mathbf{r}'|' \neq |\mathbf{r}''|$.

- v' is called the *tangential component of acceleration*, since it determines how much the acceleration vector is in the direction of \mathbf{T} . In 3D, we have

$$v' = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

- κv^2 is called the *normal component of acceleration*, since it determines how much the acceleration vector is in the direction of \mathbf{N} . In 3D, we have

$$\kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

2 3D Tangents

2.1 Partial Derivatives, §14.3

A *partial derivative* of $f(x, y, z)$ with respect to y , notated f_y or $\partial f / \partial y$, is obtained by imagining x and z are constant and taking the derivative as usual. Repeated partials are possible: f_{xy} means first take the partial with respect to x and then with respect to y . $\frac{\partial^2 f}{\partial y \partial x}$ means the same thing.

- Typically, $f_{xy} = f_{yx}$. This is true for twice continuously differentiable functions.

2.2 Tangent Planes and Approximations, §14.4

Let $f(x, y)$ be a function of two variables.

- The *tangent plane* to f at a point (x_0, y_0, z_0) (i.e. $z_0 = f(x_0, y_0)$) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This is the unique plane passing through (x_0, y_0, z_0) with the same partial derivatives as f at (x_0, y_0) .

- The *tangent plane approximation* is the assumption that $z \approx f$ in the above formula, for (x, y) near (x_0, y_0) . (Here \approx means “is approximately equal to”.)
- The *linear approximation* generalizes the tangent plane approximation. Given $g(x, y, \dots)$ and a point $P = (x_0, y_0, \dots)$, it says

$$g(x, y, \dots) \approx g(P) + g_x(P)(x - x_0) + g_y(P)(y - y_0) + \dots$$

where $(x, y, \dots) \approx P$.

- The *differential of $f(x, y)$ at (x_0, y_0)* makes computing linear approximations more convenient. It is

$$df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy.$$

df estimates the change in f in going from (x_0, y_0) to $(x_0 + dx, y_0 + dy)$ using the tangent plane approximation. It generalizes to more variables by adding more terms. Sometimes one can calculate f_x, f_y and f easily at some initial point (x_0, y_0) , while computing f exactly at a point close by is not easy. The differential lets you at least approximate the value of f at this new point through an easy computation.

2.3 Two Variable Maximums and Minimums, §14.7

Let $f(x, y)$ be a function of two variables defined on a region R . R has an “interior” and a “boundary” (possibly empty). For instance, the rectangle $[0, 1] \times [2, 3]$ has interior $(0, 1) \times (2, 3)$ and a boundary made up of four line segments. In general an interior point is one in which you can go in any direction some small amount and stay within R , while a boundary point is one where this is not the case.

- A *critical point* is a point of the interior of R where $f_x = f_y = 0$ or where one of these does not exist. You generally find critical points by solving these equations.
- On the interior of R , a *local maximum* or *local minimum* (i.e., a *local extremum*) occurs at a critical point of f . Not every critical point is a local maximum or minimum.
- To find local extremums on the interior of R , find the critical points of f and apply the The Second Derivatives Test: if (x, y) is a critical point, then...

– Compute

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

at (x, y) .

- If $D > 0$ and $f_{xx} > 0$, then (x, y) is a *local minimum*.
- If $D > 0$ and $f_{xx} < 0$, then (x, y) is a *local maximum*.
- If $D < 0$, then (x, y) is a *saddle point*, which is neither a local minimum nor a local maximum.
- Otherwise, the test gives no information.
- To find absolute extrema of f on R ,
 - Find the critical points of f .
 - Compute f at each critical point. (You do not need to classify the type of critical point.)
 - Maximize f on the boundary:
 - * Break up the boundary into curves.
 - * Parameterize each curve.
 - * Find the absolute extrema of f on each curve, typically by computing f at (single variable) critical points and boundary points.
 - Compare all values of f you’ve computed. The largest/smallest are the absolute maximum(s)/minimum(s).

3 Integration with Multiple Variables

3.1 Double Integrals, §15.1

Pick $f(x, y)$ and a region R in the xy -plane. In §15.1, R is a rectangle $[a, b] \times [c, d]$, but this is not essential.

- The *double integral of f over R* is

$$\iint_R f(x, y) dA.$$

It represents the volume below the graph of f and above the region R in the xy plane, except if f is negative the volume may be negative.

- A double integral can sometimes be computed geometrically (eg. for cylinders). It can also be approximated with *double Riemann sums*.
- The *average value of f over R* is

$$\frac{1}{A(R)} \iint_R f(x, y) dA,$$

where $A(R)$ is the area of R .

3.2 Iterated Integrals over Rectangles, §15.2

Pick $f(x, y)$ and a rectangle $R = [a, b] \times [c, d]$ in the xy -plane.

- Fubini's Theorem allows us to compute double integrals using *iterated integrals*, namely

$$\boxed{\iint_R f(x, y) dA = \int_a^b \left[\int_c^d f(x, y) dy \right] dx = \int_c^d \left[\int_a^b f(x, y) dx \right] dy.}$$

- If $f(x, y) = g(x)h(y)$, then

$$\iint_R g(x)h(y) dA = \left(\int_a^b g(x) dx \right) \left(\int_c^d h(y) dy \right).$$

3.3 Iterated Integrals over Regions, §15.3

Pick $f(x, y)$ and a region R in the xy -plane.

- The *area of R* is

$$A(R) = \iint_R 1 dA.$$

- The region R can be much more complicated than a rectangle, and yet a variant of Fubini's Theorem still holds. We have two important special cases:

- If R is given by x values from a to b and y values from $g(x)$ to $h(x)$, then

$$\boxed{\iint_R f(x, y) dA = \int_a^b \int_{g(x)}^{h(x)} f(x, y) dy dx}$$

For a picture, see §15.3, Figure 5.

- If R is given by y values from c to d and x values from $u(y)$ to $v(y)$, then

$$\boxed{\iint_R f(x, y) dA = \int_c^d \int_{u(y)}^{v(y)} f(x, y) dx dy}$$

For a picture, see §15.3, Figure 7.

Note: sometimes you may have to split a region up into pieces and apply the above on each piece.

- *Switching the order of integration* is where you are given an iterated integral in one of the forms above and are asked to find the integral in the other form. To do this, you
 - Draw the region you are integrating over.
 - Rewrite the limits in the other form.
 - Write down the new integral.

See WebAssign, 15.3(13) for an example.