Math 126 Midterm 1 Review Sheet
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Notes: Many of these have obvious generalizations to other dimensions. Some formulas are omitted, particularly those in §12.3 Theorem 2, §12.4 Theorem 11, and §13.2 Theorem 3. Section numbers refer to Stewart, Multivariable Calculus, 7th Edition.

1 Vectors

1.1 Basics, §12.2

- Notation: $a\mathbf{i}+b\mathbf{j}+c\mathbf{k} = \langle a, b, c\rangle = (a, b, c)$. Points are also (a, b, c); this can be thought of as a vector from the origin to this point.
- The vector from point P to point Q is $\boxed{\overrightarrow{PQ} = Q - P}$. "End minus beginning."
- Distance between points (a, b, c) and (p, q, r): $\boxed{D = \sqrt{(a-p)^2 + (b-q)^2 + (c-r)^2}}$
- $\langle x, y\rangle$ is perpendicular to $\langle y, -x\rangle$.
- Vector length: $\boxed{|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}}$
- Make a (non-zero) vector have length 1: $\boxed{\mathrm{unit}(\mathbf{v}) = \mathbf{v}/|\mathbf{v}|}$

1.2 Dot Products and Projections, §12.3

- Dot product is $\boxed{\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$.
- "Physical interpretation": $\boxed{\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta}$ where $0 \le \theta \le \pi$ is the (smaller) angle between the vectors.
- The above gives $\boxed{\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2}$, $\boxed{\theta = \cos^{-1}\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}}$, and $\boxed{\mathbf{a} \text{ is perpendular to } \mathbf{b} \Leftrightarrow \mathbf{a}\cdot\mathbf{b} = 0}$. ($\Leftrightarrow$ means "if and only if".)
- The **scalar projection of** v **onto** d is $\boxed{\mathrm{comp}_{\mathbf{d}}\mathbf{v} = \mathbf{v}\cdot\frac{\mathbf{d}}{|\mathbf{d}|} = \mathbf{v}\cdot\mathrm{unit}(\mathbf{d})}$. See Figure 5 of §12.3 for intuition. This is 0 if $\mathbf{v}$ is perpendicular to $\mathbf{d}$; $+|\mathbf{v}|$ if $\mathbf{v}$ is parallel to $\mathbf{d}$ and in the same direction; $-|\mathbf{v}|$ if $\mathbf{v}$ is parallel to $\mathbf{d}$ and in the opposite direction; and intermediate values otherwise.

- The **vector projection of** v **onto** d is $\boxed{\text{proj}_{\mathbf{d}}\mathbf{v} = (\text{comp}_{\mathbf{d}}\mathbf{v})\ \text{unit}(\mathbf{d}) = \frac{\mathbf{v}\cdot\mathbf{d}}{|\mathbf{d}|^2}\mathbf{d}}$. See Figure 4 of §12.3 for intuition.

1.3 Cross Products and Scalar Triple Products, §12.4

- Remember the cross product formula using Laplace expansion (Google it if needed):

$$\boxed{\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}}$$

  where $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

- Direction interpretation: $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ with orientation given by the right hand rule (see below).

- Magnitude interpretation: $\boxed{|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta}$, where θ is the (smaller) angle between $\mathbf{a}$ and $\mathbf{b}$. This is the (signed) area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$, where the sign is determined by the orientation of the parallelogram. Take absolute values if actual area is desired. This interpretation gives $\boxed{\mathbf{a} \text{ is parallel to } \mathbf{b} \Leftrightarrow \mathbf{a}\times\mathbf{b} = \mathbf{0}}$

- The right-hand rule determines the orientation of $\mathbf{a}\times\mathbf{b}$. Put your right thumb along $\mathbf{a}$, put your remaining fingers along $\mathbf{b}$, and your palm will point in the direction of $\mathbf{a}\times\mathbf{b}$.

- From the right-hand rule, one finds $\boxed{\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})}$

- The torque about a point P where a force $\mathbf{F}$ is applied at point Q is $\boxed{\tau = \mathbf{r}\times\mathbf{F}}$, where $\mathbf{r}$ points from P to Q. See Example 6 of §12.4.

- Scalar triple product: given by

$$\boxed{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}$$

  Physically, this is the (signed) volume of the parallelpiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. Take absolute value if actual volume is desired.

2 Parametrics

2.1 Line Representations, §12.5

Let $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ be any point on the line and let $\mathbf{v} = \langle a, b, c \rangle$ be a *direction vector* for the line, that is, any non-zero vector parallel to the line.

- Vector form: $\boxed{\mathbf{r}(t) = \mathbf{r_0} + t\mathbf{v}}$.
- Parametric form: $\boxed{x(t) = x_0 + at, y(t) = y_0 + bt, z(t) = z_0 + ct}$.
- Symmetric form (be careful not to divide by 0): $\boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$.
- The distance from a point (x, y, z) to the x-axis is $\boxed{\sqrt{y^2 + z^2}}$. The other axes are similar. Other point-line formulas are somewhat complicated; see the MathWorld article "Point-Line Distance–3-Dimensional" or exercise 45 of §12.4.

2.2 Plane Representations, §12.5

- Standard form 1: $\boxed{ax + by + cz = d}$. Here the *normal vector* $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular to the plane and entirely determines the plane's orientation. d is a constant giving a measure of how far from the origin the plane is.
- Standard form 2: $\boxed{ax + by + cz + d = 0}$.
- Normal/point form: $\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$ for a, b, c as above and $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ is any point on the plane.
- Vector form: $\boxed{\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}}$ where $\mathbf{r} = \langle x, y, z \rangle$ and the rest are as above.
- Using Standard form 2, the distance from a point $P = (x_1, y_1, z_1)$ to the plane is

$$\boxed{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

2.3 Parametric curves, §10.1-10.2, 13.1-13.2

- A *parametric curve* (in 3D) is $\boxed{\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle}$.
- If $\mathbf{r}(t)$ is thought of as position, then $\boxed{\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle}$ is velocity. That is, $\mathbf{r}'(t)$ gives the direction vector of the tangent line to the curve $\mathbf{r}(t)$ at the point t, and the magnitude of $\mathbf{r}'(t)$ is the speed at that point.

- Integrals work for parametric curves; for instance they can recover position vectors from velocity vectors. They are done component-wise:

$$\boxed{\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \rangle.}$$

  Indefinite integrals also work but now the $+\mathbf{c}$ is a vector, i.e. there are different constants for each component.

- For 2D parametric curves (§10.2):
  - The slope at a point is given by $\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$.
  - The second derivative is $\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) / \frac{dx}{dt} = \frac{x'y''-x''y'}{(x')^3}}$.

    Note: this is *not* the same as $\left(\frac{d^2y}{dt^2}\right) / \left(\frac{d^2x}{dt^2}\right)$.
  - If $\frac{d^2y}{dx^2} > 0$, we say the curve is *concave up*. If $\frac{d^2y}{dx^2} < 0$, we say the curve is *concave down*.

2.4 Arc Length and Curvature, §13.3

- The *arc length* of a (3D) parametric curve is $\boxed{\int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt}$
- A curve $\mathbf{r}(t)$ is *parameterized by arc length* when the arc length from 0 to t is t. This occurs if and only if it has speed 1, i.e. $|\mathbf{r}'(t)| = 1$ for all t.
- The *unit tangent* is given by $\boxed{\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)|}$.
- The curvature of a curve is κ. It has several definitions:
  - *If the curve is parametrized by arc length*, then $\kappa(s) = |\mathbf{T}'(s)|$
  - *For arbitrary parametrizations*, $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)|$
  - *For arbitrary parametrizations in 3D*, $\boxed{\kappa(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)|/|\mathbf{r}'(t)|^3}$
  - *For graphs* $y = f(x)$, $\kappa(x) = |f''(x)|/[1 + (f'(x))^2)]^{3/2}$

2.5 Polar Coordinates, §10.3

- Given a point P at distance r from the origin and at angle θ measured from the positive x-axis with more positive angles going counterclockwise, P is at position $\boxed{(r\cos\theta, r\sin\theta)}$. (r,θ) are the *polar coordinates* for the point P.
- A *polar curve* is a function of the form $\boxed{r = f(\theta)}$, where these are polar coordinates.

- The slope of a polar curve is given by

$$\boxed{\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}}$$

  For more on this, see §10.3, Example 9.

3 Quadric Surfaces, §12.6

A **quadric surface** is a 2D surface in 3D space defined by the solutions of $f(x, y, z) = 0$ for some three-variable polynomial f where each term has degree at most 2. There are only "essentially" the following quadric surfaces.

Note: See Table 1 of §12.6 for pictures.

- Sphere, center (x_0, y_0, z_0), radius r: $\boxed{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2}$.
- Ellipsoid: axis parameters (a, b, c): $\boxed{x^2/a^2 + y^2/b^2 + z^2/c^2 = 1}$.
- Elliptic paraboloid: $\boxed{z/c = x^2/a^2 + y^2/b^2}$. To remember: $x = 0$ trace is parabola $z/c = y^2/b^2$, $y = 0$ trace is also parabola, but $z = c$ trace is ellipse $1 = x^2/a^2 + y^2/b^2$.
- Hyperbolic paraboloid: $\boxed{z/c = x^2/a^2 - y^2/b^2}$. To remember: $x = 0$ and $y = 0$ traces are parabolas, but $z = c$ trace is hyperbola $1 = x^2/a^2 - y^2/b^2$.
- Cone: $\boxed{z^2/c^2 = x^2/a^2 + y^2/b^2}$. To remember: $z = k$ traces are ellipses; z^2 controls radii of ellipse, and radius increases linearly with z, giving a cone.
- Hyperboloid of one sheet: $\boxed{x^2/a^2 + y^2/b^2 - z^2/c^2 = 1}$. To remember: just one minus sign; the $z = 0$ trace is an ellipse; the $x = 0$ and $y = 0$ traces are hyperbolas.
- Hyperboloid of two sheets: $\boxed{-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1}$. To remember: two minus signs; the $z = 0$ trace is empty since $-x^2/a^2 - y^2/b^2 = 1$ has no solutions; the $x = 0$ and $y = 0$ traces are hyperbolas.
- Cylinder: in this course, start with a curve in a plane and draw lines perpendicular to the plane through each point of the curve to make a surface. The result is a (generalized) *cylinder*. The type of curve (eg. parabola) determines the type of cylinder. A usual cylinder could be called a circular cylinder, since the plane curve is a circle.

4 Identities

$$\boxed{\cos^2\theta + \sin^2\theta = 1},$$

$$\boxed{\cos 2\theta = \cos^2\theta - \sin^2\theta}, \qquad \boxed{\cos^2\theta = \tfrac{1+\cos 2\theta}{2}}.$$

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}, \qquad \boxed{\sin^2\theta = \tfrac{1-\cos 2\theta}{2}}.$$