Math 126 Final Review Sheet

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1 Material Review

See Midterm 1 and 2 reviews. Topics not covered in those reviews are below.

1.1 Double Integrals in Polar Coordinates (§15.4)

Given a function f and a region R, the Cartesian integral of f over R is

$$\iint_R f(x, y)\, dA.$$

If R can be easily expressed using polar coordinates (for instance, if R is the intersection of certain circles), this integral can be converted to polar as follows:

(1) Replace $f(x, y)$ with $f(r\cos(\theta), r\sin(\theta))$.

(2) Replace dA with $r\, dr\, d\theta$.

(3) Find appropriate limits of integration which describe R using polar coordinates. For instance, the unit circle would have limits $\theta = [0, 2\pi]$ and $r = [0, 1]$. This step typically requires some creativity or geometric intuition; at a minimum you generally need to draw R. See also Example 3 of §15.4.

Indeed, given a Cartesian integral of the above form, one can draw the limits of integration in the xy-plane and convert the integral to polar. This is sometimes useful for evaluating integrals, and you may be asked to do this on the exam. See WebAssign 15.4(12).

1.2 Center of Mass (§15.5)

You are given a shape represented by a region D, and you are also given the density $\rho(x, y)$ of that shape at each point of D.

- The **mass** is $\boxed{m = \iint_D \rho(x, y)\, dA}$.
- The **moment about the x-axis** is $M_x = \iint_D y\rho(x, y)\, dA$.
- The **moment about the y-axis** is $M_y = \iint_D x\rho(x, y)\, dA$.

- The **center of mass** is the point $(\overline{x}, \overline{y})$ given by

$$\boxed{\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\,dA, \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\,dA.}$$

  Note: ρ could be doubled and the center of mass would not change. In general, we only need to know ρ up to a multiplicative constant (eg. "proportional to" something) to determine the center of mass; the constant will cancel when dividing M_y or M_x and m.

It's often convenient to evaluate these integrals using polar coordinates.

2 Taylor Series Notes

2.1 Taylor Polynomials and Taylor's Inequality, §1-3

Using integration by parts repeatedly, one can show that for any fixed b and any positive integer n

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n + \frac{1}{n!}\int_b^x f^{(n+1)}(t)(x-t)^n\,dt$$

This formula is complicated, so we hope it's powerful (it is). Here $f^{(n)}(b)$ means the nth derivative of f at b, $n!$ means $n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, with $0! = 1$ by convention.

The terms before the integral make up the **nth Taylor polynomial** based at b for $f(x)$:

$$\boxed{T_n(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2 + \cdots + \frac{f^{(n)}(b)}{n!}(x-b)^n}.$$

In sigma (Σ) notation, this is

$$\boxed{T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b)}{k!}(x-b)^k},$$

where we interpret $f^{(0)}$ to mean f, and we take $(x-b)^0 = 1$ (even at $x = b$).

Two cases are typically called out for special attention in this course:

$$\boxed{T_1(x) = f(b) + f'(b)(x-b)}, \qquad \boxed{T_2(x) = f(b) + f'(b)(x-b) + \frac{f''(b)}{2}(x-b)^2}$$

T_2 is called the *quadratic approximation* for f based at b, or the second Taylor polynomial. T_1 is called the *tangent line approximation* for f based at b, or the first Taylor polynomial.

T_n approximates f, and we can say how good the approximation is using the first formula in this section. Doing so gives the following:

- **Tangent line error bound**: if $|f''(t)| \le M$ for all t in a fixed interval I containing b, then for any x in I,

$$\boxed{|f(x) - T_1(x)| \le \frac{M}{2}|x-b|^2}$$

- **Quadratic approximation error bound**: if $|f'''(t)| \le M$ for all t in a fixed interval I containing b, then for any x in I,

$$\boxed{|f(x) - T_2(x)| \le \tfrac{M}{6}|x-b|^3}$$

- **Taylor's inequality**: if $|f^{(n+1)}(t)| \le M$ for all t in a fixed interval I containing b, then for any x in I,

$$\boxed{|f(x) - T_n(x)| \le \tfrac{M}{(n+1)!}|x-b|^{n+1}}$$

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2.2 Basic Taylor Series, §4

The **Taylor series** for a function $f(x)$ based at b is

$$\boxed{\sum_{k=0}^{\infty} \tfrac{f^{(k)}(b)}{k!}(x-b)^k = \lim_{n\to\infty} T_n(x)},$$

A Taylor series **converges** for some x if the limit above exists and is finite at that x. The following are our "Basic Taylor Series", which you are expected to know.

Function	Series	Converges for ...
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$	$-\infty < x < \infty$
$\cos(x)$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$	$-\infty < x < \infty$
$\sin(x)$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$	$-\infty < x < \infty$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$-1 < x < 1$

The series for $\frac{1}{1-x}$ is called the *geometric series.*

2.3 Taylor Series Manipulations, §5

You can sometimes compute the Taylor series for a complicated function out of simpler Taylor series. Here are a few tricks for doing so.

- Add series/multiply by a constant:

$$\begin{aligned} 2e^x - \frac{3}{1-x} &= 2\sum_{k=0}^{\infty} \frac{x^k}{k!} - 3\sum_{k=0}^{\infty} x^k \\ &= \boxed{\sum_{k=0}^{\infty} \left(\tfrac{2}{k!} - 3\right) x^k}. \end{aligned}$$

  While e^x converges for $-\infty < x < \infty$, $\frac{1}{1-x}$ converges only for $-1 < x < 1$. The series above then converges on the overlap, i.e. for $\boxed{-1 < x < 1}$.

- Substitution:

$$\frac{1}{2x-5} = -\frac{1}{5}\cdot\frac{1}{1-\frac{2}{5}x} = -\frac{1}{5}\sum_{k=0}^{\infty}\left(\frac{2}{5}x\right)^k$$
$$= \boxed{\sum_{k=0}^{\infty}\left(-\frac{2^k}{5^{k+1}}\right)x^k}.$$

  The series used converges for $-1 < \frac{2}{5}x < 1$, i.e. for $\boxed{-\frac{5}{2} < x < \frac{5}{2}}$.

- Term-by-term differentiation:

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty}x^k\right) = \sum_{k=0}^{\infty}\frac{d}{dx}x^k$$
$$= \sum_{k=0}^{\infty}kx^{k-1} = \sum_{k=-1}^{\infty}(k+1)x^k = \boxed{\sum_{k=0}^{\infty}(k+1)x^k}.$$

  In the second to last step, we *reindexed* the sum: we replaced k with $k+1$. The lower limit, $k=0$, then becomes $k+1=0$, i.e. $k=-1$. However, $(k+1)x^k$ for $k=-1$ is 0, so we can start the reindexed sum at $k=0$.

  $\frac{1}{1-x}$ converges for $-1 < x < 1$, and it turns out in general that differentiating term-by-term doesn't change the "interval of convergence," so the series converges for $\boxed{-1 < x < 1}$.

- Term-by-term integration:

$$\int_0^x e^{-t^2}\,dt = \int_0^x \sum_{k=0}^{\infty}\frac{(-t^2)^k}{k!}\,dt = \sum_{k=0}^{\infty}\int_0^x \frac{(-1)^k}{k!}t^{2k}\,dt$$
$$= \boxed{\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\cdot k!}x^{2k+1}} = x - \frac{x^3}{3} + \frac{x^5}{10} + \cdots.$$

  Like differentiation, this does not change the interval of convergence. Since the series for e^x converges for all x, the same is true of the series for e^{-t^2}, and hence the above is valid for $\boxed{-\infty < x < \infty}$.

- You can read off Taylor polynomials from a Taylor series. For instance, since

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

  is the Taylor series for e^x based at 0, we see $T_1(x) = 1 + x$ and $T_2(x) = 1 + x + \frac{x^2}{2}$ are the first and second Taylor polynomials for e^x based at 0.