Math 126 Challenge Problems/Solutions

Problems Posted 11/26/2013
Solutions Posted 12/3/2013

1. Use Taylor's Theorem on the arctan function to find and prove a simple infinite sum expression to compute π. Can you calculate π to 50 digits this way?

Note that $\arctan(1) = \pi/4$, so evaluating the Taylor series for $\arctan(x)$ at $x = 1$ will give a series for $\pi/4$, hence for π.

Writing the Taylor series out, we have (hopefully!)

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{\arctan^{(k)}(0)}{k!} x^k.$$

One can use Taylor's Inequality to show that the error term goes to 0 for all real x, hence the above sum is indeed correct. Conveniently, $\arctan(0) = 0$, so the first term drops out. Since $\arctan'(x) = 1/(1+x^2)$, we have $\arctan'(0) = 1$. Fiddling with the derivatives for a while, you can discover a pattern (at least when they're evaluated at 0) which gives, after some simplification,

$$\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}.$$

Plugging in $x = 1$ and multiplying by 4 gives the series

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}.$$

What if we cut off this sum at, say, $k = 100$? The Taylor's Inequality bound mentioned above can give us an idea of how close the approximation will be, which turns out to not be terribly accurate. Indeed, with $k = 100$ we get $3.15149\ldots$, which is only correct to the first decimal place.

It does turn out that if we want m correct digits, there is indeed some n such that if we cut off the sum at $k = n$ or later, we do get at least m correct digits. However, n gets very large compared to m for this series, and there are much better (but more advanced) series available, some of which can even take $n \approx m$ or better. ∎

2. (Tricky.) Is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ finite or infinite? If finite, what is its value? If infinite, prove it.

<u>Hint:</u> One method is to consider integrals. There are many solutions.

Try adding the series up on your calculator. You might think it's settling on a fixed value, but if you add enough terms it'll just keep increasing, albeit very slowly.

The full series is called the *harmonic series*; the partial sums $\sum_{k=1}^{n} \frac{1}{n}$ are called *harmonic numbers*. Proofs that the harmonic series sums to infinity are well-known. Wikipedia's article on this series mentions the two most popular proofs, which happen to be the main ones I'm familiar with.

(1) First, the extremely clever, very elementary one:

$$\begin{aligned}1+&\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots\\&>1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots\\&=1+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\dots=\infty.\end{aligned}$$

(2) Second, the integral test: using left Riemann sums, one can show that the given sum is at least as large as $\int_1^\infty \frac{1}{x}\,dx$, which, using the logarithm function, is infinite. This proof is perhaps less desirable in that it requires knowledge of the logarithm function, though it very strongly suggests the following approximation:

$$\sum_{k=1}^{n}\frac{1}{k}\approx\ln(n).$$

Indeed, this is a good approximation, and the difference $\left(\sum_{k=1}^{n}\frac{1}{k}\right)-\ln(n)$ happens to converge to a finite number, γ, the "Euler–Mascheroni constant" or sometimes just "Euler's Constant"; it happens to be approximately $0.577\ldots$. The legendary mathematician Leonhard Euler ("Oiler") first described the constant in 1735 and by 1781 he had calculated it to 16 digits. (Note: Euler was blind during the last few decades of his life, including when this calculation was published. He was astonishing in many ways.)

(3) A student's solution used the Taylor series for $-\ln(1-x)$, namely $-\ln(1-x)=\sum_{k=1}^{\infty}\frac{1}{k}x^k$. Taking $x=1$ in this expression gives $+\infty=\sum_{k=1}^{\infty}\frac{1}{k}$, proving divergence–except this isn't a proof since there are plenty of missing details. What we would like is the following:

$$\begin{aligned}+\infty=-\ln(0)&=\lim_{x\to1^-}-\ln(1-x)=\lim_{x\to1^-}\sum_{k=1}^{\infty}\frac{1}{k}x^k\\&=\sum_{k=1}^{\infty}\lim_{x\to1^-}\frac{1}{k}x^k=\sum_{k=1}^{\infty}\frac{1}{k}.\end{aligned}$$

The first and second equalities come from basic properties of ln. The third equality uses the Taylor series for $-\ln(1-x)$; this step needs justification, though one can use the error bound from our Taylor Series notes to find that the error bound tends to zero as the number of terms in the Taylor series tends to ∞, which justifies it. The fourth equality is the hard part. It follows from the following:

Theorem 1 (Abel's Limit Theorem (Special Case)) *Let a_k for $k=0,1,2,\ldots$ be a sequence of non-negative real numbers. If $\sum_{k=0}^{\infty}a_k<\infty$, then*

$$\lim_{x\to1^-}\sum_{k=0}^{\infty}a_kx^k=\sum_{k=0}^{\infty}\lim_{x\to1^-}a_kx^k=\sum_{k=0}^{\infty}a_k.$$

You can find proofs of this theorem in many places. The ones I'm familiar with make essential use of an estimate obtained from the geometric series formula from an earlier challenge problem.

To wrap up the above proof, note we can only apply Abel's Limit Theorem if the harmonic series converges, but it diverges. Still, we have a proof by contradiction: assuming the harmonic series converged, the above string of equalities holds, so $+\infty=\sum_{k=1}^{\infty}\frac{1}{k}$, contradicting our assumption that the harmonic series converged. So, it diverges, as claimed!

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