Math 126 Challenge Problems/Solutions

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1. Suppose $f(x)$ has a continuous derivative. Prove the following.

Theorem 1 (Mean Value Theorem) *Fix $a < b$. The slope of the line connecting $f(a)$ and $f(b)$ is*

$$\frac{f(b) - f(a)}{b - a}.$$

There is some point c between a and b where the derivative of f at c agrees with the slope of the above line. □

Suppose the theorem fails for some function f and some points a, b. That is, suppose for each point c between a and b, $f'(c) \neq (f(b) - f(a)/(b - a)$. Consider the difference $D(c) = f'(c) - (f(b) - f(a))/(b - a)$, which has $D(c) \neq 0$ by assumption. Since f' is continuous, $D(c)$ is continuous. If $D(c)$ is positive for some c_1 and negative for some c_2, then from the intermediate value theorem there is a point c between c_1 and c_2 where $D(c) = 0$, contrary to our assumption.

So, D is always positive or always negative—equivalently, f' is either always above or always below the line's slope. But intuitively this can't happen: the function would lie strictly above or strictly below the line connecting them at every point between a and b. Formally,

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(c)\,dc = \int_a^b \left(D(c) + \frac{f(b) - f(a)}{b - a} \right) dc \\ &= \int_a^b D(c)\,dc + f(b) - f(a). \end{aligned}$$

However, since $D(c) > 0$ or $D(c) < 0$, $\int_a^b D(c)\,dc$ is > 0 or < 0, even though it must be 0 once we cancel the $f(b) - f(a)$'s from both sides, a contradiction. This proves the theorem. Note that we haven't given an algorithm for finding the point c in the theorem statement, we've just shown that if no such c exists, one of the basic assumptions about real numbers is broken. ■

2. Use the Mean Value Theorem to prove

$$e^x > 1 + x$$

for $x > 0$.

Let $a = 0$, $b = x$ in the Mean Value Theorem, with $f(x) = e^x$. From that theorem, there is a point c (strictly) between 0 and x such that $f'(c) = e^c = (f(x) - f(0))/(x - 0) = (e^x - 1)/x$. Rearranging, $xe^c = e^x - 1$, or $xe^c + 1 = e^x$. Now $c > 0$, so $e^c > 1$, so $1 + x < 1 + xe^c = e^x$, giving the stated inequality. ■