Math 126 Challenge Problems/Solutions

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1. Find a formula for changing from cartesian to "elliptic" coordinates $x = r\cos(\theta), y = 2r\sin(\theta)$ akin to $\iint_R f(x,y), dA = \iint_T f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$ for polar coordinates. (Note that in these coordinates $x^2 + (y/2)^2 = r^2$, hence my choice of name.)

The main trouble is determining what to replace dA with; we're essentially forced to replace R with T', which is a set of points with elliptic coordinates (r, θ) corresponding to our Cartesian points R. We also need to replace $f(x, y)$ with $f(r\cos(\theta), 2r\sin(\theta))$. To find what dA becomes, imagine we had a small rectangle with opposite vertexes (r, θ) and $(r + dr, \theta + d\theta)$. After converting to Cartesian, we get a sector of an ellipse. We can compute the area by just scaling the y-axis down by a factor of 2, since that turns the ellipse into a circle, and the proof of the Cartesian to polar formula shows that for small $dr, d\theta$, the area of the resulting circular sector is $r\, dr\, d\theta$. We then must replace dA with $2r\, dr\, d\theta$ to go to elliptic coordinates.

This is part of a much more general procedure one may use to compute formulas like this. Let $\Phi(u, v) = (x(u, v), y(u, v))$ be a change of coordinates, going from some coordinate system (u, v) to Cartesian coordinates (x, y). Now compute the following determinant:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

This turns out to be the coefficient of the new $du\, dv$. In our case, it is

$$\begin{vmatrix} \frac{\partial r\cos(\theta)}{\partial r} & \frac{\partial r\cos(\theta)}{\partial \theta} \\ \frac{\partial 2r\sin(\theta)}{\partial r} & \frac{\partial 2r\sin(\theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ 2\sin(\theta) & 2r\cos(\theta) \end{vmatrix} = 2r(\cos^2(\theta) + \sin^2(\theta)) = 2r,$$

as expected. (Of course, not just any Φ works, but that's a discussion for another time.) ∎

2. Compute $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ exactly. (Hint: consider double integrals.)

We can integrate xe^{-x^2} by substitution, but not e^{-x^2}. Still, the exponent is part of $-x^2 - y^2 = -r^2$ in polar, and when switching to polar we pick up an extra factor of r. That is,

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy &= \int_0^{2\pi}\int_0^{\infty} re^{-r^2}\, dr\, d\theta \\ &= 2\pi \left.\frac{e^{-r^2}}{-2}\right|_{r=0}^{\infty} = \pi. \end{aligned}$$

But on the other hand, we have

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\, dx\, dy \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2. \end{aligned}$$

Comparing these computations and taking a square root gives

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.}$$

This integral is called the Gaussian integral. ■