Math 126 Challenge Problems/Solutions

Problems Posted 11/12/2013 Solutions Posted 11/14/2013

1. Compute the surface area of a sphere using Thursday, November 7th's challenge problem #2 and polar coordinates.

The referenced formula for the surface area of the graph of $f(x, y)$ over a region R in the xy-plane is

$$S = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA.$$

For $f(x, y) = \sqrt{1 - x^2 - y^2}$, we compute

$$\begin{aligned} f_x &= -\frac{x}{\sqrt{1 - x^2 - y^2}} \\ \Rightarrow \sqrt{1 + (f_x)^2 + (f_y)^2} &= \sqrt{1 + \frac{x^2 + y^2}{1 - x^2 - y^2}} \\ &= \frac{1}{\sqrt{1 - x^2 - y^2}}. \end{aligned}$$

In polar coordinates, our integrand is then simply $\frac{1}{\sqrt{1-r^2}}$. Using polar integration with the region R given by the unit disk, we have $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 1$, hence (letting $u = 1 - r^2$)

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 (1 - r^2)^{-1/2} r\, dr\, d\theta \\ &= 2\pi \int_1^0 -\frac{u^{-1/2}}{2}\, du \\ &= \pi 2u^{1/2}|_{u=0}^1 \\ &= 2\pi. \end{aligned}$$

This is just the surface area of the top half of the sphere, so the total surface area is $\boxed{4\pi}$, as expected. ■

2. Can you compute the surface area of the ellipsoid $x^2 + (y/2)^2 + z^2 = 1$ with the method from (1)? What about the paraboloid $z = x^2 + y^2$ for $x^2 + y^2 \leq 1$?

(The ellipsoid question was accidentally not asked in class; instead the paraboloid question was asked. While it is possible to do the ellipsoid integral in this special case, it is somewhat lengthy, so a full writeup is not included. Roughly, solve for y in terms of the other variables, and proceed as in (1). You'll integrate over an ellipse rather than a circle, though it's still doable. It turns out that if all three axes differ, the problem is extremely difficult; there is no general solution in terms of standard functions in that case.)

For the paraboloid, let $f(x, y) = x^2 + y^2$, so $\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2} = \sqrt{1 + 4r^2}$ in polar. The integral is then

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^1 r\sqrt{1 + 4r^2}\, dr\, d\theta \\ &= \ldots = \boxed{\tfrac{\pi}{6}(5\sqrt{5} - 1)}. \end{aligned}$$

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