Math 126 Challenge Problems/Solutions

Problems Posted 11/5/2013 Solutions Posted 11/7/2013

1. Find a function for which the conclusion of Fubini's theorem does not hold.

The statement of Fubini's theorem from the text is as follows:

Theorem 1 (Fubini) *If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy.$$

More generally this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist. □

Our function must then violate some of these conditions. In this question we're left very free to violate these conditions; the next question imposes more constraints. Here is a relatively simple construction. Let $R = [0, 1] \times [0, 1]$ and define $f(x, y)$ as follows. Set...

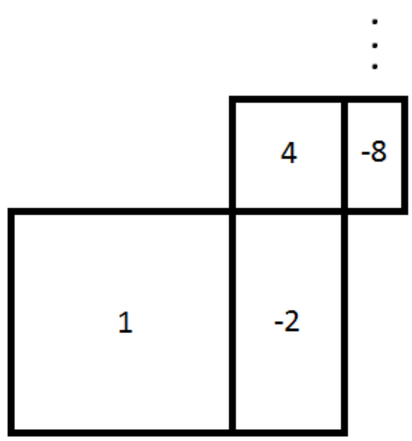

Figure 1: Fubini counterexample $f(x, y)$ on the unit square $[0, 1] \times [0, 1]$.

- $f = 1$ on $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$.
- $f = -2$ on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4}] \times [0, \frac{1}{2}]$.
- $f = 4$ on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{4}] \times [\frac{1}{2}, \frac{1}{2} + \frac{1}{4}]$.
- $f = -8$ on $[\frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}] \times [\frac{1}{2}, \frac{1}{2} + \frac{1}{4}]$.
- etc.

Figure 1 illustrates the basic idea and pattern. A small note: I've defined f in two ways at $(\frac{1}{2}, 0)$, first as 1 and second as -2. Since the boundary strips are "thin", this doesn't matter, though if we wanted to be careful we could just pick one definition. I'll blithely ignore this double definition and leave a careful argument to you.

Now, compute the two iterated integrals. Doing vertical strips, we find that if $0 \leq x \leq \frac{1}{2}$ we get $\int_0^1 f(x,y)\,dy = \frac{1}{2}$. For $\frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{4}$, the -2 and the 4 cancel, giving $\int_0^1 f(x,y)\,dy = 0$. Similarly all remaining x-values give 0 integral, so

$$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{4}$$

On the other hand, integrating in horizontal strips gives cancellation every single time, so

$$\int_0^1 \int_0^1 f(x,y)\,dx\,dy = 0$$

. So, f is such a counterexample. Why didn't Fubini's theorem work? Mostly, f is both arbitrarily large (positive) and arbitrarily small (negative) at certain points. f is also discontinuous on all the boundary strips, but since they effectively contribute nothing to the integral, this isn't actually a problem. Two more remarks:

(1) What happens if we stop the above construction after the $f = -8$ definition above, taking $f = 0$ elsewhere? Fubini's theorem applies, and is even true—the -8 contribution to the vertical strip integrals cancels the $\frac{1}{2}$ we got earlier. We really do need to carry out the construction "to infinity".

(2) What's going on here is related to the concept of rearranging infinite sums. It turns out that, given an infinite sum $\sum_{n=0}^{\infty} a_n$ for constants a_n, it's possible to choose a_n so wickedly that if we add them up in a different order, we can get the resulting sum to converge to any value we wish. Changing the order of integration is analogous to rearranging a badly behaved (double) sum. (For more on this, Google "Riemann Rearrangement Theorem".)

■

2. Find a continuous function (continuous on the interior of the region of integration) for which the iterated integral is a finite value but switching the order of integration gives a different finite value.

What was essential to the construction in (1) working? We needed a point for which the function is both arbitrarily large (positive) and arbitrarily small (negative) near that point, and we need the integral of these positive and negative parts to each be infinite.

As a first try, we need our function to blow up at a point, say the origin, so we might try $f(x,y) = 1/(x+y)$ on $[0,1] \times [0,1]$. You can check this doesn't work—you can see that since f is perfectly symmetric in x and y, the iterated integrals cannot possibly differ. So, we need some asymmetry and also some negativity; maybe we could try $f(x,y) = (x-y)/(x+y)$. This is better but it doesn't blow up at the origin, it's merely discontinuous there: approach the origin from the $+y$ axis to get -1, approach from the $+x$ axis to get 1.

As an aside, we can compute the integral of this most recent f using Fubini's theorem. Since the function doesn't blow up, the theorem quoted above applies, so interchanging x and y gives the same integral, i.e.

$$\int_0^1 \int_0^1 \frac{x-y}{x+y}\,dx\,dy = \int_0^1 \int_0^1 \frac{x-y}{x+y}\,dy\,dx.$$

However, we can swap the symbols "x" and "y" in the right-hand side, which gives the left-hand side multiplied by -1. This forces both sides to be 0.

Now we want the denominator to head to zero quickly enough for the entire expression to blow up at the origin. If we just square the denominator, i.e. use $f(x,y) = (x-y)/(x+y)^2$, this indeed occurs. Moreover, approaching the origin from the $+x$ axis gives a limit of $+\infty$ while approaching from the $+y$ axis gives a limit of $-\infty$, so we're zeroing in on the properties of the function in (1). Unfortunately, the integral of the positive and negative parts are each finite—one can integrate $|f(x, y)| = |x - y|/(x + y)^2$ to see this (one gets $2 - \ln 4$).

We need the denominator to go to zero even more quickly so that the function gets larger faster as we head to the origin so the positive and negative areas are infinite. Finally, one can verify that $f(x, y) = (x - y)/(x + y)^3$ solves the problem: $dx\,dy$ gives $-1/2$ and $dy\,dx$ gives $1/2$.

Note: these integrals might be tricky to evaluate. I had a computer algebra system do them so I don't know their difficulty. For $f(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$, the Wikipedia article on Fubini's Theorem has a lengthy calculation section. The example I derived above also happens to be Exercise 39 of §15.2 of Stewart, though we arrived at this solution independently. ■