Math 126 Challenge Problems/Solutions

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1. Let $p(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial with complex number coefficients c_k, where z is a complex number. Show that the real and imaginary parts of p, viewed as functions of x and y where $z = x + iy$, are harmonic functions, i.e. they satisfy Laplace's equation, $u_{xx} + u_{yy} = 0$.

We first verify this for $p(z) = z^k$. Using the chain rule,

$$\begin{aligned} \frac{\partial}{\partial x}(x+iy)^k &= k(x+iy)^{k-1}\frac{\partial}{\partial x}(x+iy) = k(x+iy)^{k-1} \\ \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(x+iy)^k\right) &= \ldots = k(k-1)(x+iy)^{k-2} \\ \frac{\partial}{\partial y}(x+iy)^k &= k(x+iy)^{k-1}\frac{\partial}{\partial y}(x+iy) = k(x+iy)^{k-1}i \\ \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(x+iy)^k\right) &= \ldots = k(k-1)(x+iy)^{k-2}i^2. \end{aligned}$$

Since $1 + i^2 = 1 - 1 = 0$, adding these gives 0, so Laplace's equation is satisfied. (Here we've implicitly used the fact that, given $p(x + iy) = u(x, y) + iv(x, y)$ where u, v are the real and imaginary parts of p, respectively, $\frac{\partial}{\partial x}p = \frac{\partial}{\partial x}u + i\frac{\partial}{\partial y}$, and that the chain rule works as one would expect.)

Next, note that we can multiply a harmonic function by a real number without changing the fact that it satisfies Laplace's equation. We can also add two harmonic functions and get a harmonic function similarly. Altogether, $c_k z^k$'s real and imaginary components are harmonic for each k, so their sum, $p(z)$, has harmonic real and imaginary components. ∎

2. A version of the *minimum modulus principle* says that if the real and imaginary components of a complex function $f(z)$ are harmonic, and the function is never 0, then an absolute minimum of $|f(z)|$ on a bounded domain occurs at a boundary point. Use this and (1) to prove the Fundamental Theorem of Algebra, that every non-constant polynomial $p(z)$ has at least one complex root.

Suppose to the contrary that we had some non-constant polynomial $p(z)$ with no complex root. From (1), its real and imaginary components are harmonic, so we can apply the minimum modulus principle—this is where we use the assumption that p is never zero. Consider the domain of p to be the circle of radius R, $C_R = \{x + iy \mid x^2 + y^2 \leq R^2\}$. We have some point z_R on the boundary of C_R such that $|p(z_R)| \leq |p(z)|$ for all $|z| \leq R$—in words, z_R gives $|p(z)|$ its minimum value on C_R.

However, say $p(z)$ has highest-degree term $c_n z^n$, where $n \geq 1$ since p is assumed non-constant. Now for $|z|$ extremely large, the other terms of the polynomial contribute negligibly to $|p(z)|$. Indeed, for $|z|$ large enough, $|p(z)| \geq |c_n z^n|/2$. For z_R, this gives $|p(z_R)| \geq |c_n|R^n/2$. But since $|p(z_R)| \leq |p(z)|$ for all $|z| \leq R$, we have

$$|p(z)| \geq |c_n|R^n/2, \qquad \text{if } |z| \leq R$$

This is true for every R; taking $R \to \infty$ forces $|p(z)| = \infty$ for all z, which is a contradiction. So, $p(z)$ must have had a root after all. (We can recover the usual Fundamental Theorem of Arithmetic, that p has n roots counting multiplicity, by repeating this argument on $p(z)/(z - r)$ where r is a root, etc.) ∎