Math 126 Challenge Problems/Solutions

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1. Given $f(x)$ smooth, find a polynomial which agrees with f at 0 and has the same first n derivatives as f at 0. That is, find p such that $p^{(k)}(0) = f^{(k)}(0)$ for $0 \leq k \leq n$. ("Smooth" means f has derivatives of all orders at all points.)

Say $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ for some constants c_k that we'll have to determine. Since $p(0) = c_0$, we need $c_0 = f(0)$ to get the 0th derivatives to match up. Since $p'(x) = c_1 + 2c_2 x + \cdots$, we have $p'(0) = c_1$, so we need $p'(0) = c_1 = f'(0)$. Similarly $p''(x) = 2c_2 + x(\cdots)$ forces $p''(0) = 2c_2 = f''(0)$. Repeating this, one finds in general the condition to get the kth derivative of p to agree with the kth derivative of f at 0 is

$$p^{(k)}(0) = k! c_k = f^{(k)}(0),$$

where $k! = k \times (k-1) \times \ldots \times 1$ is the factorial function, and for convenience we say $0! = 1$. Solving for c_k and substituting, the polynomial is just

$$p(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n. \quad \square$$

Strictly speaking, we could add higher order terms like x^{2n} to the end without messing up the first n derivatives' values at 0, so there are infinitely many answers. However, the 0th through nth coefficients are uniquely determined by the given condition. ■

2. Now take $f(x, y)$ smooth. Find a polynomial in x and y such that f and p agree up to second partials at the origin, i.e. the following hold at $(0, 0)$:

- $f = p$
- $f_x = p_x$, $f_y = p_y$
- $f_{xy} = p_{xy}$, $f_{xx} = p_{xx}$, $f_{yx} = p_{yx}$, $f_{yy} = p_{yy}$

("Smooth" means f has partial derivatives of all orders at all points.)

Say

$$p(x, y) = c_{0,0} + c_{1,0}x + c_{0,1}y + c_{1,1}xy + c_{2,0}x^2 + c_{0,2}y^2 + \cdots$$

We have $p(0, 0) = c_{0,0}$. One can compute

$$p_x = c_{1,0} + c_{1,1}y + 2c_{2,0}x + (\cdots),$$

where every term in $(\cdots)$ has at least one x or y in it. So, $p_x(0, 0) = c_{1,0}$. Similarly, one can compute the other partials in terms of just the first six constants listed above:

$$p_y(0,0) = c_{0,1}$$
$$p_{xx}(0,0) = 2c_{2,0}$$
$$p_{xy}(0,0) = c_{1,1} = p_{yx}(0,0)$$
$$p_{yy}(0,0) = 2c_{0,2}$$

Our polynomial is then

$$p(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + f_{xy}(0,0)xy + \frac{f_{xx}(0,0)}{2}x^2 + \frac{f_{yy}(0,0)}{2}y^2.$$

It happens that we can write this nicely using matrices (and dropping the $(0,0)$ from the notation after the partials):

$$p(x,y) = f(0,0) + \begin{pmatrix} f_x & f_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(Expand it out and check for yourself!)

- The row vector in the middle term is called the "Jacobian" and determines the behavior of f "up to first order". (In this particular case, the Jacobian and the "gradient" are the same thing.) The Jacobian appears in the multivariable change of variables formula; we'll encounter a special case of this soon.

- The matrix in the right term is called the "Hessian" and determines the behavior of f "up to second order". It appears when more careful approximation than just using the Jacobian is needed.

  The Hessian appears in the "Second Derivatives Test". This is no coincidence. Suppose $(0,0)$ is a critical point of f, so $f_x = f_y = 0$ at $(0,0)$. Since we only want to know if $(0,0)$ is a local max, min, or saddle of p, we may just as well assume $f(0,0) = 0$. Our polynomial is then

  $$p(x,y) = \frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

  Applying a little linear algebra involving eigenvalues and eigenvectors to this expression gives precisely the statement of the Second Derivatives Test.

  With more careful reasoning, one can show that the behavior of p in this regard is the same as the behavior of the function f it approximates. A careful treatment of these ideas requires an estimate of just how well p approximates f near $(0,0)$. At the end of the quarter, we'll do some estimation of this form for functions of a single variable.

Again one can add more higher order monomials to the end of p above, so there are technically an infinite number of solutions. ∎