Math 126 Challenge Problems/Solutions

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1. Show that a curve $\mathbf{r}(t)$ whose tangent vectors at each point are perpendicular to $\mathbf{r}(t)$ is contained in a sphere centered at the origin.

We have $\mathbf{r}'(t)$ perpendicular to $\mathbf{r}(t)$, i.e. $\mathbf{r}'(t) \cdot \mathbf{r}(t)$. But we can use the product rule for dot products to compute $d/dt(|\mathbf{r}(t)|^2) = d/dt(\mathbf{r}(t) \cdot \mathbf{r}(t)) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$, so $|\mathbf{r}(t)|$ is constant. That is, $\mathbf{r}(t)$ is contained in a sphere centered at the origin. ■

2. Let $f(x, y) = (x + iy)^2$. Show that

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 2(x + iy).$$

(Identifying z with both (x, y) and $x + iy$, we may write $f(z) = z^2$, and the right-hand side of the above is $2z$.)

Since $(x + iy)^2 = (x^2 - y^2) + i(2xy)$, we compute

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) &= \frac{1}{2}\left(\frac{\partial}{\partial x}[(x^2 - y^2) + i(2xy)] - i\frac{\partial}{\partial y}[(x^2 - y^2) + i(2xy)]\right) \\
&= \frac{1}{2}\left([2x + i(2y)] - i[-2y + i(2x)]\right) \\
&= \frac{1}{2}\left(2x + i(2y) + i(2y) + 2x\right) \\
&= 2(x + iy).
\end{aligned}$$

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