Math 126 Challenge Problems/Solutions

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1. Parameterize an "L"-shaped curve in the plane as follows:

$$\mathbf{r}(t) = \begin{cases} (t,0) & \text{if} \quad -1 \le t \le 0 \\ (0,t) & \text{if} \quad 0 \le t \le 1 \end{cases}$$

Graphically, this curve starts at $(-1,0)$, moves right at speed 1 until it hits the origin, suddenly turns straight north, and stops when it hits $(0,1)$. Note that $\mathbf{r}'(t)$ exists for $-1 < t < 1$ except for at $t = 0$, when the curve turns. Find a reparameterization of $\mathbf{r}$ such that the first derivative exists everywhere.

(For our purposes, a *reparameterization* of $\mathbf{r}(t)$ is something of the form $\mathbf{r}(f(s))$, for some non-decreasing function f, which traces out the same points as $\mathbf{r}$. Sometimes $\mathbf{r}(f(s))$ is written as $\mathbf{r}(s)$.)

Since we can't change the direction of $\mathbf{r}'(t)$ by a reparameterization, we have to scale $\mathbf{r}'(t)$ somehow in order to get the derivative to exist at $t = 0$. In particular, if we shrink $\mathbf{r}'(t)$ to length zero as t approaches 0, the derivative will exist. Explicitly, such a parameterization is given by $t(s) = s^3$, i.e.

$$\mathbf{r}(s) = \mathbf{r}(t(s)) = \begin{cases} (s^3,0) & \text{if} \quad -1 \le s \le 0 \\ (0,s^3) & \text{if} \quad 0 \le s \le 1 \end{cases}$$

since

$$\mathbf{r}'(s) = \begin{cases} \langle 3s^2,0\rangle & \text{if} \quad -1 \le s \le 0 \\ \langle 0,3s^2\rangle & \text{if} \quad 0 \le s \le 1 \end{cases}$$

has $d\mathbf{r}/ds|_{s=0} = \mathbf{0}$ on both pieces. Indeed, $\mathbf{r}''(s = 0) = \mathbf{0}$ as well, but $\mathbf{r}'''(s = 0)$ is $\langle 6,0\rangle$ for the first piece and $\langle 0,6\rangle$ for the second piece, so the third derivative does not exist even with this parametrization.

(Note: Some sources would not quite accept this as a reparameterization, since they would require the change of variables function to have a differentiable inverse, whereas here the change of variables function has derivative 0 at a single point (so the inverse has "derivative" ∞ there). The key property that the integral of the original curve is the same as the integral of this curve is preserved, however, so in my mind a slightly more general definition is best.) ■

2. (Hard) Find a reparametrization as in (1) but where $\mathbf{r}^{(n)}(s)$, the nth derivative of $\mathbf{r}$ at s, exists for all s and for all n.

In general, a reparametrization is given by an increasing function $t(s)$. In (1) we used $t(s) = s^3$, which is indeed increasing at every point. Note that $t'(s) = 3s^2$ has $t'(0) = 0$ and $t''(0) = 0$, at which points $\mathbf{r}'$ and $\mathbf{r}''$ existed since they were forced to be $\mathbf{0}$ on either piece. However, $t'''(0) = 6 \ne 0$, and $\mathbf{r}'''$ did not exist at 0. This suggests we want $t(s)$ such that $t^{(n)}(0) = 0$ for <u>all</u> integers $n \ge 1$ and $t(s)$ increasing. Suppose for the moment we have such a function, where also $t(0) = 0$ for convenience. We can compute the derivativse of the reparameterized curve at $s = 0$ and verify that it is indeed 0 on each piece; here's the first derivative verification, which is just the chain rule:

$$\begin{aligned}\left.\frac{d\mathbf{r}(t(s))}{ds}\right|_{s=0} &= \left.\frac{d\mathbf{r}(t)}{dt}\right|_{t=t(0)} \left.\frac{dt(s)}{ds}\right|_{s=0} \\ &= \left.\frac{d\mathbf{r}(t)}{dt}\right|_{t=0} 0 = \mathbf{0}.\end{aligned}$$

(Strictly speaking, $d\mathbf{r}(t)/dt|_{t=0}$ does not exist, but we may apply the above reasoning rigorously to each piece of $\mathbf{r}(t)$ and get $\mathbf{0}$ from both sides, so the overall derivative is in fact $\mathbf{0}$.) The general verification is the same but more involved; it uses the generalized chain rule, which looks something like the binomial theorem.

Now, how do we construct the function $t(s)$? No polynomial will work since a degree n polynomial has non-zero (constant) nth derivative. This may suggest we want a function going to zero "exponentially quickly". Explicitly, look at $t(s) = e^{-1/s^2}$ (define $t(0) = 0$). If you plot this function, it looks like an inverted bell curve–it is extremely flat near $s = 0$ where it hits 0, so it probably has the required $t^{(n)}(0) = 0$ property. We needed an increasing function $t(s)$, and this isn't increasing, but that's easy to fix–just flip the left half over the x-axis. This would give us a piecewise function, but a virtually equivalent approach is to just look at $t(s) = se^{-1/s^2}$. A graph of this shows it to definitely be increasing (rigorously, it's easy to show that the first derivative is positive everywhere except 0). One can use the product rule to show that if e^{-1/s^2} has the "derivatives-are-0-at-0" property, so does this $t(s)$. So, it suffices to show that e^{-1/s^2} indeed has derivatives all equal to 0 at $s = 0$.

If you compute a few derivatives of t, the following proposition is plausible:

Proposition 1 *There are polynomials p_n and q_n such that*

$$t^{(n)}(s) = \frac{p_n(s)}{q_n(s)} e^{-1/s^2}$$

Proof We use "mathematical induction". Suppose you have an infinite number of statements to prove, one for each value of n. You can prove them all by proving two things: (1) the 1st statement is true; (2) *if* the nth statement is true, *then* the $n+1$st statement is true. For instance, given (1) and (2), the 3rd statement is true since the 1st was true by (1), so by (2) the 2nd is true, so by (2) again the 3rd is true.

We prove these two pieces here. We actually start with $n = 0$, which is fine.

(1) Base case: For $n = 0$, the polynomials $p_0 = q_0 = 1$ work.

(2) Inductive case: Suppose the proposition is true for n. From the product, quotient, and chain rules, we have

$$\begin{aligned}t^{(n+1)}(s) = \frac{d}{ds}t^{(n)}(s) &= \frac{d}{ds}\frac{p_n(s)}{q_n(s)}e^{-1/s^2} \\ &= \frac{2p_n(s)}{s^3 q_n(s)}e^{-1/s^2} + \frac{q_n(s)p_n'(s) - p_n(s)q_n'(s)}{q_n^2(s)}e^{-1/s^2}\end{aligned}$$

Finding a common denominator would give explicit formulas for p_{n+1} and q_{n+1}, though existence is all we need, so we stop here. □

We can't just plug in $s = 0$ to the above expressions (even if we could compute them quickly) since we would divide by zero. If the limit as $s \to 0$ of the above exists, though, one may show the derivative exists there and is the limiting value. (See "Darboux's Theorem" and discussions of discontinuities of derivatives.)

To simplify matters, set $1/s^2 = y$, i.e. take $s = y^{-1/2}$ for $y > 0$. In light of the above proposition and discussion, we now wish to compute

$$\lim_{y\to\infty} \frac{u_n(y)}{v_n(y)} e^{-y}$$

where u_n and v_n are polynomials in $\sqrt{y}$. (Check this yourself.)

Intuitively, the exponential factor should go to zero quickly enough to overwhelm the rate at which the polynomial might go to infinity, since exponential decay beats polynomial growth. More rigorously, we have...

Proposition 2 *For any real number k (positive, negative, etc.),*

$$\lim_{y\to\infty} y^k e^{-y} = 0$$

Proof If $k < 0$, both terms tend to 0 anyway. If $k = 0$, $y^k = 1$ and the result again follows. In the remaining case, write the limit as $\lim_{y\to\infty} y^k/e^y$. This gives the indeterminate form ∞/∞, so apply L'Hopital's Rule:

$$\lim_{y\to\infty} \frac{y^k}{e^y} = \lim_{y\to\infty} \frac{ky^{k-1}}{e^y}$$

While y^{k-1} may still go to infinity, repeating this will eventually decrease the exponent on y to a non-positive number, which we decided already was alright, giving the proposition. Induction can formalize this. □ ■

The denominator $v_n(y)$ is either a constant or gets very large as $y \to \infty$, since it's a polynomial, so we can ignore it. The numerator u_n for y large enough is dominated by its highest-degree term, say y^k, so $|u_n(y)| < 2y^k$. The limit we wished to compute is then 0 by Proposition 0.2. We restricted ourselves to $y > 0$ here, which used $s > 0$, so we've actually computed

$$\lim_{s\to 0^+} t^{(n)}(s) = 0$$

But the original function t is even, from which it follows that

$$\lim_{s\to 0^-} t^{(n)}(s) = 0$$

Since the left and right limits exist and agree, it follows that

$$t^{(n)}(0) = \lim_{s\to 0} t^{(n)}(s) = 0,$$

which completes the proof. ■