Math 126 Challenge Problems/Solutions

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1. (Hard.) A *cardiod* is given parametrically by

$$x(\theta) = \cos\theta + \frac{1}{2}\cos 2\theta, \quad y(\theta) = \sin\theta + \frac{1}{2}\sin 2\theta.$$

Find a Cartesian equation for this curve. (For instance, if $x(\theta) = \cos\theta, y(\theta) = \sin\theta$, the Cartesian equation would be $x^2 + y^2 = 1$.)

There are many solutions, but here is one. For convenience, define some auxiliary variables: $u = \cos\theta$, $v = \sin\theta$. From the double angle formulas for cos and sin we have $x = u + (1/2)(u^2 - v^2)$, $y = v + (1/2)(2uv) = v(1+u)$. From the Pythagorean Theorem, $u^2 + v^2 = 1$, so $u^2 - v^2 = 2u^2 - 1$, giving $x = u^2 + u - 1/2$. Our goal is to find a relationship between x and y which doesn't involve u and v. If we square y, we can eliminate v: $y^2 = v^2(1+u)^2 = (1-u^2)(1+u)^2$. If we multiply this out, we get $1 + 2u - 2u^3 - u^4$. It would be nice to eliminate the u^4 term, which we can do by adding x^2, since it involves u^4: we compute $x^2 = (u^2 + u - 1/2)^2 = 1/4 - u + 2u^3 + u^4$. Now we have

$$x^2 + y^2 = 5/4 + u, \tag{1}$$

so u has almost been eliminated. We have x in terms of only u, so we can hopefully replace $5/4 + u$ with something involving x. We can't quite do this directly, but we do have $x = u^2 + u - 1/2 = (u + 1/2)^2 - 3/4$, i.e. $x + 3/4 = (u + 1/2)^2$. We can subtract $3/4$ from both sides of (1) and square the result to get $(x^2 + y^2 - 3/4)^2 = (u + 1/2)^2 = x + 3/4$. That is, our final relation is

$$(x^2 + y^2 - 3/4)^2 = x + 3/4. \tag{2}$$

You can graph this curve and find that it indeed is the cardioid.

Note: There is strictly speaking more to prove. Points on the cardioid must satisfy (2), but might there be points satisfying (2) that are not on the cardioid? Indeed, depending on how one does the above derivation, you might get a more complicated final relation that includes extra points. However, it turns out points on (2) are indeed precisely the points of the cardioid. The proof of this is beyond the scope of this writeup, though, since it is easy enough to have a computer graph the solution set and check it is just the cardioid. ■

2. Plot some of the following odd-looking polar curves and find an interesting one of your own! (Stewart, §13.3(67-72).)

(a) $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

(b) $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

(c) $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

(d) $r = |\tan\theta|^{|\cot\theta|}$ (valentine curve)

(e) $r = 1 + \cos^{999}\theta$ (PacMan curve)

(f) $r = \sin^2(4\theta) + \cos(4\theta)$

(You can type these into Wolfram Alpha directly to plot them.) ■