Math 126 Challenge Problems/Solutions

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1. Given $z = f(x, y)$, a *tangent plane* at (x_0, y_0) is a plane containing

   (i) the point (x_0, y_0, z_0) (where $z_0 = f(x_0, y_0)$) and

   (ii) the tangent lines through (x_0, y_0, z_0) for the two traces $x = x_0$, $y = y_0$, i.e. the lines tangent to the intersection of the surface and the planes $x = x_0$, $y = y_0$ at (x_0, y_0, z_0).

Compute the tangent plane at $(0, 0)$ for $f(x, y) = x^2 + y^2$.

The trace of $x^2 + y^2 = z$ for $x = 0$ is $y^2 = z$. The tangent line to $y^2 = z$ at $y = 0$ is horizontal, i.e. in the yz-plane it has direction vector $\langle 1, 0\rangle$. To lift this up to three dimensions, set the x-coordinate to 0, giving the line $t\langle 0, 1, 0\rangle = t\mathbf{j}$, i.e. the y-axis. In virtually the same way, the $x = 0$ trace gives us the line $s\mathbf{i}$, i.e. the x-axis. We need a plane containing the origin and these lines; certainly the xy-plane is the only solution.

$x^2 + y^2 = z$ is an elliptic (circular) paraboloid oriented with the circles along the z-axis. If you draw a picture, it should be clear that the tangent plane at the origin is indeed the xy-plane, by symmetry. ■

2. Compute the tangent plane to $z = x^2 + y^2$ at each point (x_0, y_0).

The $x = x_0$ trace is $x_0^2 + y^2 = z$, which is still a parabola in the yz-plane. The line tangent to (y_0, z_0) has direction vector

$$\left\langle 1, \frac{d}{dy}(x_0^2 + y^2)\Big|_{y=y_0} \right\rangle.$$

This is because the derivative gives us the slope of the tangent line in the yz-plane at the point in question. As above, to lift this line up to a line in 3D space, we set the x-component of the direction vector to 0, i.e. we get $v_x = \langle 0, 1, 2y_0\rangle$. For $y = y_0$, the same procedure gives us $v_y = \langle 1, 0, 2x_0\rangle$.

Now, we need a plane passing through $(x_0, y_0, x_0^2 + y_0^2)$ containing lines with direction vectors v_x and v_y. The normal vector is just

$$v_x \times v_y = \langle 2x_0, 2y_0, -1\rangle,$$

so the equation of the plane is

$$2x_0(x - x_0) + 2y_0(y - y_0) - (z - z_0) = 0.$$

This simplifies a bit: $-2x_0^2 - 2y_0^2 + z_0 = -z_0$. Thus the equation of the plane is

$$2x_0x + 2y_0y = z + z_0.$$

At $x = y = 0$, this indeed gives $0 = z$, the xy-plane, as above.

Note: The explicit form of $f(x, y)$ was almost entirely irrelevant to the above derivation. The same argument shows that, for general $f(x, y)$, the tangent plane at (x_0, y_0) is

$$\frac{d}{dx}f(x, y_0)\Big|_{x=x_0}(x - x_0) + \frac{d}{dy}f(x_0, y)\Big|_{y=y_0}(y - y_0) = z - z_0.$$

The unwieldy derivatives are called *partial derivatives*, which we'll get to later in the course. In one standard notation, the above becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = z - z_0.$$

This will be a very important expression towards the end of the course.

(One small note: we have assumed that f is not too "badly behaved", eg. that the derivatives we've taken exist. The exact conditions needed for tangent planes to make sense is a topic for later courses.) ∎