Math 126 Challenge Problems/Solutions

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Solutions Posted 10/10/2013

1. Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are 3D vectors. They form a *spanning set* if every 3D vector is of the form $a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, i.e. we can reach any point by starting at the origin and moving in the three directions $\mathbf{u}, \mathbf{v}, \mathbf{w}$. Determine when the vectors $\langle 1, p, p^2\rangle, \langle 1, q, q^2\rangle, \langle 1, r, r^2\rangle$ form a spanning set.

[Hint: Use 10/03 Challenge Problems, #1.]

Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ forms a spanning set. Consider the parallelpiped they determine. Geometrically, it is clear that the volume of the parallelpiped is non-zero, since otherwise the vectors would be trapped in a plane and one couldn't reach points off the plane by traveling only in the directions $\mathbf{u}, \mathbf{v}, \mathbf{w}$. If they don't form a spanning set, then the volume must be zero similarly. Thus $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a spanning set if and only if the associated parallelpiped has non-zero volume. From the referenced problem, the parallelpiped associated to the given vectors has volume $(p-q)(q-r)(r-p)$. This is non-zero if and only if each of p, q, r is distinct. In all, the given vectors form a spanning set if and only if p, q, r are each distinct. ■

2. Suppose two quadratic polynomials agree at three distinct points. Using the previous problem, show that the polynomials are in fact equal.

Let the polynomials be

$$g(x) = a + bx + cx^2$$
$$h(x) = d + ex + fx^2.$$

Suppose they agree at distinct points p, q, r, i.e. $g(p) = h(p)$, etc. Since $g(p) - q(p) = 0$, etc., we have the following system of equations:

$$(a-d) + (b-e)p + (c-f)p^2 = 0$$
$$(a-d) + (b-e)q + (c-f)q^2 = 0$$
$$(a-d) + (b-e)r + (c-f)r^2 = 0.$$

This can be analyzed using vectors. Let $\mathbf{A} = \langle a-d, b-e, c-f\rangle$, $\mathbf{u} = \langle 1, p, p^2\rangle$, $\mathbf{v} = \langle 1, q, q^2\rangle$, and $\mathbf{w} = \langle 1, r, r^2\rangle$. The system is then just

$$\mathbf{A}\cdot\mathbf{u} = 0$$
$$\mathbf{A}\cdot\mathbf{v} = 0$$
$$\mathbf{A}\cdot\mathbf{w} = 0.$$

Now, from the previous problem, $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a spanning set. In particular, $\mathbf{A}$ is of the form $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}$ for some scalars α, β, γ. Using the linearity of the dot product (see §12.3, Theorem 2) and the above three relations, we have

$$\begin{aligned}\mathbf{A}\cdot\mathbf{A} &= \mathbf{A}\cdot(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w})\\ &= \alpha(\mathbf{A}\cdot\mathbf{u}) + \beta(\mathbf{A}\cdot\mathbf{v}) + \gamma(\mathbf{A}\cdot\mathbf{w})\\ &= \alpha 0 + \beta 0 + \gamma 0\\ &= 0.\end{aligned}$$

Thus, $\mathbf{A}$ is the zero vector, so $a - d = b - e = c - f = 0$, i.e. $a = d, b = e, c = f$, so $g(x) = h(x)$ are in fact equal. ∎